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Quantum-invariant theory and the evolution of a Dirac field in Friedmann–Robertson–Walker flat space-times

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Abstract. On the basis of the generalized invariant formulation, the invariant-related unitary transformation method is used to study the evolution of a quantum Dirac field in Friedmann–Robertson–Walker spatially flat space-times. We first solve the functional Schrödinger equation for a free Dirac field and obtain the exact solutions. We then investigate the way of extending the method to treat the case in which there is an interaction between the Dirac field and a scalar field.

1 Introduction

There has been a sustained effort to study the dynamics of the quantum scalar field in Friedmann–Robertson–Walker (FRW) flat space-times because of its intimate connection with various problems in modern cosmology (see [1] and references therein). The quantum Dirac field dynamics in FRW flat space-times is also relevant to some of the cosmological problems, such as that of the decay into Dirac particles by scalar particles produced at the preheating stage in inflationary cosmology [4].

In order to study this decay of scalar particles into Dirac particles at the preheating stage, the dynamics of a Dirac field interacting with a scalar field in FRW flat space-times should be investigated. There are many standard calculations concerning the dynamics of the Dirac fields in curved space [3,5,2,6], but to our knowledge, no one has ever given a general formulation in the literature for the treatment of a quantum Dirac field interacting with a quantum scalar field in FRW flat space-times. It is the purpose of this paper to develop such a general formulation.

The Hamiltonian for a quantum Dirac field in FRW flat space-times has explicit time dependence. The Schrödinger picture of quantum field theory affords a relatively clear viewpoint from which to understand many important aspects of the quantum fields with time-dependent Hamiltonians and has been developed with some success in recent years [1,7–13]. In this paper, we will work in the Schrödinger picture and use the quantum-invariant theory to study a Dirac field interacting with a scalar field in FRW flat space-times.

The quantum-invariant theory was first proposed by Lewis and Riesenfeld [14] and then generalized by us [15]. On the basis of this generalized formulation, the invariant-related unitary transformation method was developed and used to treat some time-dependent quantum scalar fields and Dirac fields [11,12,16]. In [11], we obtained the complete set of the exact solutions of the time-dependent functional Schrödinger equation for the free scalar field in FRW space-times; with the help of the complete set of the exact solutions, we found a way of extending the formulation to treat the case in which there is an interaction; in [12], we treated a quantum Dirac field in a time-dependent homogeneous electric field. In the present paper, we will use the formulation to do the same thing for the Dirac field interacting with a scalar field in FRW flat space-times.

The present paper is organized as follows. In Sect. 2, the invariant-related unitary transformation method is used to obtain the complete set of the exact solutions of the time-dependent functional Schrödinger equation for the free Dirac field in FRW flat space-times. In Sect. 3, the complete set obtained in Sect. 2 is used to treat the Dirac field interacting with a scalar field. In Sect. 4, there are some concluding remarks. Finally, in Appendix Appendix A., we briefly review the invariant formulation and the invariant-related unitary transformation method, and in Appendix Appendix B., we present the auxiliary equations which are much more complicated than that obtained in [11] and can only be solved numerically.

2 Evolution of a free quantum Dirac field in FRW spatially flat space-times

The (1+3)-dimensional FRW flat metric is of the form

$$\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \qquad (\mu,\nu = 0,1,2,3)\,, \tag{2.1}$$
$$g_{00} = 1, g_{ij} = -a^2(t)\,\delta_{ij} \qquad (i,j = 1,2,3)\,,$$

where $\mathrm{d}x^0 = \mathrm{d}t, \mathrm{d}x^i$ is the contravariant 3-dimensional vector, and $a(t)$ is the cosmic-scale factor. The Hubble parameter is defined to be

$$h(t)=\frac{\dot a(t)}{a(t)}. \tag{2.2}$$

A free Dirac field ψ of mass m propagating in $(1+3)$-dimensional FRW flat space-times is described classically by the action ($\hbar=c=1$)

$$S=\int \mathrm{d}^{1+3}xL, \tag{2.3}$$

with

$$L(x)=[-g(t)]^{\frac{1}{2}} \times\left\{\frac{\mathrm{i}}{2}L^{\mu a}\left[\bar\psi\gamma_a\nabla_\mu\psi-\left(\nabla_\mu\bar\psi\right)\gamma_a\psi\right]-m\bar\psi\psi\right\} \tag{2.4}$$

and

$$[-g(t)]^{\frac{1}{2}}=[|\det(g_{\mu\nu})|]^{\frac{1}{2}}=[a(t)]^3 \tag{2.5}$$
$$\gamma_0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \gamma_k=\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix},$$
$$k=1,2,3$$

in which σ_k are the Pauli spin matrices, and $L_{\mu a}$ ($\mu,a=0,1,2,3$) is the vierbein field:

$$L_{\mu a}=\begin{pmatrix}1 & 0\\ 0 & a(t)\,\delta_{ij}\end{pmatrix}, \tag{2.6}$$
$$L_\mu{}^a L_{\nu a}=g_{\mu\nu},$$
$$L_{\mu a}L^\mu{}_b=\eta_{ab}=(1,-1,-1,-1),$$

and

$$\nabla_\mu\psi=(\partial_\mu+\mathrm{i}B_\mu)\psi,\quad \nabla_\mu\bar\psi=\partial_\mu\bar\psi-\mathrm{i}\bar\psi B_\mu, \tag{2.7}$$

B_μ being the spinor connection. The spinor connection is defined by

$$B_\mu=\frac{1}{4}L_{\beta c}L^\beta_{a;\mu}\sigma^{ac} \tag{2.8}$$
$$L^\beta_{a;\mu}=\partial_\mu L^\beta{}_a+\Gamma^\beta_{\mu\alpha}L^\alpha{}_a$$
$$\sigma^{ac}=\frac{\mathrm{i}}{2}[\gamma^a,\gamma^c]$$

in which $\Gamma^\beta_{\mu a}$ is the affine connection and can be obtained from the metric $g_{\mu\nu}$. We calculate the spinor connection B_μ and obtain

$$B_0=0 \tag{2.9}$$
$$B_i=\frac{1}{2}\sigma^{0\mathrm{i}}\dot a$$

Substitution of (2.5) and (2.7) into (2.5) leads to

$$L(x)=[-g(t)]^{\frac{1}{2}}\left\{\bar\psi(x,t)\left[\mathrm{i}\gamma_0\left(\frac{\partial}{\partial t}+u(t)\right)+\mathrm{i}a^{-1}(t)(\gamma_1\partial_x+\gamma_2\partial_y+\gamma_3\partial_z)-m\right]\psi(x,t)\right\} \tag{2.10}$$

where

$$u(t)=\frac{3}{2}\frac{\dot a(t)}{a(t)}, \tag{2.11}$$
$$\bar\psi=\psi^+\gamma_0.$$

By making use of (2.11), it is easy to obtain the canonical momentum density π_α

$$\pi_\alpha=\frac{\partial L}{\partial\dot\psi_\alpha}=\mathrm{i}a^3(t)\,\psi^+_\alpha. \tag{2.12}$$

To quantize this field, equal-time anticommutation relations are introduced among the operators $\hat\pi$ and $\hat\psi$

$$\left[\hat\psi_\alpha(\vec x,t),\hat\pi_\beta(\vec x',t)\right]_+=\mathrm{i}\delta_{\alpha\beta}\delta(\vec x-\vec x') \tag{2.13}$$
$$\left[\hat\psi_\alpha(\vec x,t),\hat\psi_\beta(\vec x',t)\right]_+=0,\quad [\hat\pi_\alpha(\vec x,t),\hat\pi_\beta(\vec x',t)]_+=0.$$

We choose to work within the functional Schrödinger picture [7–12]. From (2.11) and (2.12), we get the quantum time-dependent Hamiltonian for the system in the Schrödinger picture,

$$\begin{aligned}H(t)&=\int\left\{\hat\pi(\vec x,t_0)\left[-a^{-1}(t)\,\alpha_i\frac{\partial}{\partial x^i}-\mathrm{i}\beta m+u(t)\right]\hat\psi(\vec x,t_0)\right\}\mathrm{d}^3\vec x\\ &=\int\left\{a^3(t_0)\,\hat\psi^+(\vec x,t_0)\left[-\mathrm{i}a^{-1}(t)\,\alpha_i\frac{\partial}{\partial x^i}+\beta m+\mathrm{i}u(t)\right]\hat\psi(\vec x,t_0)\right\}\mathrm{d}^3\vec x\end{aligned} \tag{2.14}$$

where

$$\alpha_i=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}, \tag{2.15}$$
$$\beta=\gamma_0.$$

Because the spatial sections are flat, we can employ the momentum representation for the operators [9],

$$\hat\psi(\vec x,t_0)=\sum_{\pm s}\int\frac{\mathrm{d}^3\vec p}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E_p}}\times\left[\hat b_s(\vec p)\,u_s(\vec p)\,\mathrm{e}^{\mathrm{i}\vec p\cdot\vec x}+\hat d_s^+(\vec p)\,v_s(\vec p)\,\mathrm{e}^{-\mathrm{i}\vec p\cdot\vec x}\right] \tag{2.16}$$
$$\hat\psi^+(\vec x,t_0)=\sum_{\pm s}\int\frac{\mathrm{d}^3\vec p}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E_p}}\left[\hat b_s^+(\vec p)\,\bar u_s(\vec p)\,\gamma_0\mathrm{e}^{-\mathrm{i}\vec p\cdot\vec x}+\hat d_s(\vec p)\,\bar v_s(\vec p)\,\gamma_0\mathrm{e}^{\mathrm{i}\vec p\cdot\vec x}\right]$$

where $E_p=\sqrt{|\vec p|^2+m^2}$, $u_s(\vec p),v_s(\vec p),\bar u_s(\vec p),\bar v_s(\vec p)$ are the spinors defined in [17], $\hat b_s^+(\vec p)\left(\hat b_s(\vec p)\right)$ is the creation (annihilation) operator of an "electron" with momentum $\vec p$ and spin s, and $\hat d_s^+(\vec p)\left(\hat d_s(\vec p)\right)$ is the creation (annihilation) operator of a "positron" with $\vec p$ and s. It follows from

(2.13) that creation and annihilation operators satisfy the anticommutation relations

$$\begin{aligned}\left[\hat{b}_s(\vec{p}),\hat{b}^+_{s'}(\vec{p}')\right]_+ &= a(t_0)^{-3}\delta_{ss'}\delta(\vec{p}-\vec{p}'),\\ \left[\hat{d}_s(\vec{p}),\hat{d}^+_{s'}(\vec{p}')\right]_+ &= a(t_0)^{-3}\delta_{ss'}\delta(\vec{p}-\vec{p}')\\ \left[\hat{b}_s(\vec{p}),\hat{b}_{s'}(\vec{p}')\right]_+ &= \left[\hat{d}_s(\vec{p}),\hat{d}_{s'}(\vec{p}')\right]_+\\ &= \left[\hat{b}^+_s(\vec{p}),\hat{b}^+_{s'}(\vec{p}')\right]_+\\ &= \left[\hat{d}^+_s(\vec{p}),\hat{d}^+_{s'}(\vec{p}')\right]_+ = 0\\ \left[\hat{b}_s(\vec{p}),\hat{d}^+_{s'}(\vec{p}')\right]_+ &= \left[\hat{b}^+_s(\vec{p}),\hat{d}^+_{s'}(\vec{p}')\right]_+\\ &= \left[\hat{b}_s(\vec{p}),\hat{d}_{s'}(\vec{p}')\right]_+\\ &= \left[\hat{b}^+_s(\vec{p}),\hat{d}_{s'}(\vec{p}')\right]_+ = 0.\end{aligned} \tag{2.17}$$

Inserting (2.17) in (2.14), we get

$$\begin{aligned}\hat{H}(t) = \int \mathrm{d}^3\vec{p}\Bigg\{&\sum_{\pm s}\left[E_b(t)\hat{b}^+_s\hat{b}_s - E_d(t)\hat{d}_s\hat{d}^+_s\right]\\ &+\lambda_1(t)\hat{b}^+_{+s}\hat{d}^+_{+s} + \lambda_2(t)\hat{b}^+_{-s}\hat{d}^+_{-s}, +\lambda_3(t)\hat{b}^+_{+s}\hat{d}^+_{-s}\\ &+\lambda_4(t)\hat{b}^+_{-s}\hat{d}^+_{+s} + \lambda^*_1(t)\hat{d}_{+s}\hat{b}_{+s} + \lambda^*_2(t)\hat{d}_{-s}\hat{b}_{-s}\\ &+\lambda^*_3(t)\hat{d}_{-s}\hat{b}_{+s} + \lambda^*_4(t)\hat{d}_{+s}\hat{b}_{-s}\Bigg\}\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}&E_b(t) = a^3(t_0)\left[E_p + \mathrm{i}u(t)\right]\\ &E_d(t) = a^3(t_0)\left[E_p - \mathrm{i}u(t)\right]\\ &\lambda_1(t) = a^3(t_0)\left[a^{-1}(t)-1\right]p_z, \quad \lambda_2(t) = -\lambda_1(t)\\ &\lambda_3(t) = a^3(t_0)\left[a^{-1}(t)-1\right](p_x - \mathrm{i}p_y), \quad \lambda_4(t) = \lambda^*_3(t).\end{aligned} \tag{2.19}$$

The time-dependent functional Schrödinger equation for the system is

$$\mathrm{i}\frac{\partial}{\partial t}\Psi\left[\psi(\vec{p});t\right] = \hat{H}(t)\Psi\left[\psi(\vec{p});t\right]. \tag{2.20}$$

It is easy to show that there exists an invariant $\hat{I}(t)$ satisfying

$$\frac{\partial\hat{I}(t)}{\partial t} - \mathrm{i}\left[\hat{I}(t),\hat{H}(t)\right] = 0 \tag{2.21}$$

which is found to be

$$\begin{aligned}\hat{I}(t) = \sum_{\pm s}\int \mathrm{d}^3\vec{p}&\left[\hat{B}^+_s(\vec{p},t)\hat{B}_s(\vec{p},t)\right.\\ &\left.-\hat{D}_s(\vec{p},t)\hat{D}^+_s(\vec{p},t)\right],\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}\hat{B}_{+s}(\vec{p},t) = &\left[\cos\theta_1\cos\theta_2\cos\theta_6\right.\\ &\left.+\sin\theta_1\sin\theta_3\sin\theta_6\mathrm{e}^{-\mathrm{i}(\phi_1+\phi_3+\phi_6)}\right]\hat{b}_{+s}(\vec{p})\\ &+\left[\cos\theta_1\sin\theta_2\sin\theta_5\mathrm{e}^{\mathrm{i}(\phi_2+\phi_5)}\right.\\ &\left.-\sin\theta_1\cos\theta_3\cos\theta_5\mathrm{e}^{-\mathrm{i}\phi_1}\right]\hat{b}_{-s}(\vec{p}),\\ &+\left[\cos\theta_1\sin\theta_2\cos\theta_5\mathrm{e}^{\mathrm{i}\phi_2}\right.\\ &\left.+\sin\theta_1\cos\theta_3\sin\theta_5\mathrm{e}^{-\mathrm{i}(\phi_1+\phi_5)}\right]\hat{d}^+_{+s}(\vec{p})\\ &-\left[\cos\theta_1\cos\theta_3\sin\theta_6\mathrm{e}^{-\mathrm{i}\phi_6}\right.\\ &\left.-\sin\theta_1\sin\theta_3\cos\theta_6\mathrm{e}^{-\mathrm{i}(\phi_1+\phi_3)}\right]\hat{d}^+_{-s}(\vec{p}),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\hat{B}_{-s}(\vec{p},t) = &-\left[\cos\theta_1\sin\theta_3\sin\theta_6\mathrm{e}^{-\mathrm{i}(\phi_3+\phi_6)}\right.\\ &\left.-\sin\theta_1\cos\theta_2\cos\theta_6\mathrm{e}^{\mathrm{i}\phi_1}\right]\hat{b}_{+s}(\vec{p})\\ &+\left[\cos\theta_1\cos\theta_3\cos\theta_5\right.\\ &\left.+\sin\theta_1\cos\theta_2\cos\theta_5\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2+\phi_5)}\right]\hat{b}_{-s}(\vec{p})\\ &-\left[\cos\theta_1\cos\theta_3\sin\theta_5\mathrm{e}^{-\mathrm{i}\phi_5}\right.\\ &\left.-\sin\theta_1\sin\theta_2\cos\theta_5\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)}\right]\hat{d}^+_{+s}(\vec{p})\\ &-\left[\cos\theta_1\sin\theta_3\cos\theta_6\mathrm{e}^{-\mathrm{i}\phi_3}\right.\\ &\left.+\sin\theta_1\cos\theta_2\sin\theta_6\mathrm{e}^{\mathrm{i}(\phi_1-\phi_6)}\right]\hat{d}^+_{-s}(\vec{p}),\end{aligned}$$

$$\begin{aligned}\hat{D}^+_{-s}(\vec{p},t) = &\left[\cos\theta_4\cos\theta_3\sin\theta_6\mathrm{e}^{\mathrm{i}\phi_6}\right.\\ &\left.+\sin\theta_4\sin\theta_2\cos\theta_6\mathrm{e}^{-\mathrm{i}(\phi_2+\phi_4)}\right]\hat{b}_{+s}(\vec{p})\\ &+\left[\cos\theta_4\sin\theta_3\cos\theta_5\mathrm{e}^{\mathrm{i}\phi_3}\right.\\ &\left.-\sin\theta_4\cos\theta_2\sin\theta_5\mathrm{e}^{-\mathrm{i}(\phi_4-\phi_5)}\right]\hat{b}_{-s}(\vec{p})\\ &-\left[\cos\theta_4\sin\theta_3\sin\theta_5\mathrm{e}^{\mathrm{i}(\phi_3-\phi_5)}\right.\\ &\left.+\sin\theta_4\cos\theta_2\cos\theta_5\mathrm{e}^{-\mathrm{i}\phi_4}\right]\hat{d}^+_{+s}(\vec{p})\\ &-\left[\cos\theta_4\cos\theta_3\cos\theta_6\right.\\ &\left.-\sin\theta_4\sin\theta_2\sin\theta_6\mathrm{e}^{-\mathrm{i}(\phi_2+\phi_4+\phi_6)}\right]\hat{d}^+_{-s}(\vec{p}),\end{aligned}$$

$$\begin{aligned}\hat{D}^+_{+s}(\vec{p},t) = &-\left[\cos\theta_4\sin\theta_2\cos\theta_6\mathrm{e}^{-\mathrm{i}\phi_2}\right.\\ &\left.-\sin\theta_4\cos\theta_3\sin\theta_6\mathrm{e}^{\mathrm{i}(\phi_4+\phi_6)}\right]\hat{b}_{+s}(\vec{p})\\ &+\left[\cos\theta_4\cos\theta_2\sin\theta_5\mathrm{e}^{\mathrm{i}\phi_5}\right.\\ &\left.+\sin\theta_4\sin\theta_3\cos\theta_5\mathrm{e}^{\mathrm{i}(\phi_3+\phi_4)}\right]\hat{b}_{-s}(\vec{p})\\ &+\left[\cos\theta_4\cos\theta_2\cos\theta_5\right.\\ &\left.-\sin\theta_4\sin\theta_3\sin\theta_5\mathrm{e}^{\mathrm{i}(\phi_3+\phi_4-\phi_5)}\right]\hat{d}^+_{+s}(\vec{p})\\ &+\left[\cos\theta_4\sin\theta_2\sin\theta_6\mathrm{e}^{-\mathrm{i}(\phi_2+\phi_6)}\right.\\ &\left.+\sin\theta_4\cos\theta_3\cos\theta_6\mathrm{e}^{\mathrm{i}\phi_4}\right]\hat{d}^+_{-s}(\vec{p}),\end{aligned}$$

and $\theta_m, \phi_m, (m = 1, 2, \ldots, 6)$ are the real solutions of the auxiliary equations (see Appendix Appendix B.). It is easy to show that the operators $\hat{B}_s^+(\vec{p},t), \hat{B}_s(\vec{p},t), \hat{D}_s^+(\vec{p},t)$, $\hat{D}_s(\vec{p},t)$ satisfy the equal-time anticommutation relations

$$\left[\hat{B}_s(\vec{p},t), \hat{B}_{s'}^+(\vec{p}',t)\right]_+ = a(t_0)^{-3}\delta_{ss'}\delta(\vec{p}-\vec{p}'), \quad (2.24)$$
$$\left[\hat{D}_s(\vec{p},t), \hat{D}_{s'}^+(\vec{p}',t)\right]_+ = a(t_0)^{-3}\delta_{ss'}\delta(\vec{p}-\vec{p}').$$

According to the invariant-related unitary transformation method (see Appendix Appendix A.), the unitary transformation $\hat{V}(t)$ can be constructed:

$$\hat{V}(t) = \hat{V}_3(t)\hat{V}_2(t)\hat{V}_1(t), \quad (2.25)$$
$$\hat{V}_1(t) = \exp\int d^3\vec{p}a^3(t_0) \times\left[\left(-\theta_1 e^{-i\phi_1}\hat{b}_{+s}^+\hat{b}_{-s} + \theta_1 e^{i\phi_1}\hat{b}_{-s}^+\hat{b}_{+s}\right) + \left(-\theta_4 e^{-i\phi_4}\hat{d}_{+s}\hat{d}_{-s}^+ + \theta_4 e^{i\phi_4}\hat{d}_{-s}\hat{d}_{+s}^+\right)\right],$$
$$\hat{V}_2(t) = \exp\int d^3\vec{p}a^3(t_0) \times\left[\left(-\theta_2 e^{-i\phi_2}\hat{b}_{+s}^+\hat{d}_{+s}^+ + \theta_2 e^{i\phi_2}\hat{d}_{+s}\hat{b}_{+s}\right) + \left(-\theta_3 e^{-i\phi_3}\hat{b}_{-s}^+\hat{d}_{-s}^+ + \theta_3 e^{i\phi_3}\hat{d}_{-s}\hat{b}_{-s}\right)\right],$$
$$\hat{V}_3(t) = \exp\int d^3\vec{p}a^3(t_0) \times\left[\left(-\theta_5 e^{-i\phi_5}\hat{b}_{-s}^+\hat{d}_{+s}^+ + \theta_5 e^{i\phi_5}\hat{d}_{+s}\hat{b}_{-s}\right) + \left(-\theta_6 e^{-i\phi_6}\hat{b}_{+s}^+\hat{d}_{-s}^+ + \theta_6 e^{i\phi_6}\hat{d}_{-s}\hat{b}_{+s}\right)\right],$$

where, for simplicity, the argument $\vec{p}$ of $\hat{b}_s^+, \hat{b}_s, \hat{d}_s^+, \hat{d}_s$ is omitted. With the help of (2.17), it is easy to show that the $\hat{V}(t)$ in (2.25) transforms $\hat{I}(t)$ into the time-independent $\hat{I}_V$:

$$\hat{I}_V = \hat{V}^+(t)\hat{I}(t)\hat{V}(t) = \sum_{\pm s}\int d^3\vec{p}\left[\hat{b}_s^+(\vec{p})\hat{b}_s(\vec{p}) + \hat{d}_s(\vec{p})\hat{d}_s^+(\vec{p})\right]. \quad (2.26)$$

According to (A7), $\hat{H}_0(t)$ is defined as

$$\hat{H}_0(t) = \hat{V}^+(t)\hat{H}(t)\hat{V}(t) - i\hat{V}^+(t)\frac{\partial\hat{V}(t)}{\partial t} = \int d^3\vec{p}\hat{H}_0(\vec{p},t), \quad (2.27)$$

in which the first term is easily obtained:

$$\hat{V}^+(t)\hat{H}(t)\hat{V}(t) = \sum_{\pm s}\int d^3\vec{p}\left[\alpha_s^{(d)}(p,t)\hat{b}_s^+(\vec{p})\hat{b}_s(\vec{p}) + \beta_s^{(d)}(p,t)\hat{d}_s(\vec{p})\hat{d}_s^+(\vec{p})\right] \quad (2.28)$$

with

$$\alpha_{+s}^{(d)}(p,t) = \sin^2\theta_6\chi_4^{(d)}(p,t) + \cos^2\theta_6\chi_1^{(d)}(p,t) - \sin 2\theta_6\left[\chi_6^{(d)}(p,t)\sin\phi_6 + \chi_5^{(d)}(p,t)\cos\phi_6\right], \quad (2.29)$$
$$\alpha_{-s}^{(d)}(p,t) = \sin^2\theta_5\chi_3^{(d)}(p,t) + \cos^2\theta_5\chi_2^{(d)}(p,t) - \sin 2\theta_5\left[\chi_8^{(d)}(p,t)\sin\phi_5 + \chi_7^{(d)}(p,t)\cos\phi_5\right],$$
$$\beta_{+s}^{(d)}(p,t) = \sin^2\theta_5\chi_2^{(d)}(p,t) + \cos^2\theta_5\chi_3^{(d)}(p,t) + \sin 2\theta_5\left[\chi_8^{(d)}(p,t)\sin\phi_5 + \chi_7^{(d)}(p,t)\cos\phi_5\right],$$
$$\beta_{-s}^{(d)}(p,t) = \sin^2\theta_6\chi_4^{(d)}(p,t) + \cos^2\theta_6\chi_1^{(d)}(p,t) + \sin 2\theta_6\left[\chi_6^{(d)}(p,t)\sin\phi_6 + \chi_5^{(d)}(p,t)\cos\phi_6\right],$$

where $\chi_m^{(d)}(p,t), (m=1,2,\ldots,8)$ can be found from Appendix Appendix B.. By means of the Baker-Hausdorf–Campbell formula [18] and (2.17), the second term in (2.27) can be calculated:

$$-i\hat{V}^+(t)\frac{\partial\hat{V}(t)}{\partial t} = \sum_{\pm s}\int d^3\vec{p}\left[\alpha_s^{(g)}(p,t)\hat{b}_s^+(\vec{p})\hat{b}_s(\vec{p}) + \beta_s^{(g)}(p,t)\hat{d}_s(\vec{p})\hat{d}_s^+(\vec{p})\right], \quad (2.30)$$

with

$$\alpha_{+s}^{(g)}(p,t) = \sin^2\theta_6\chi_4^{(g)}(p,t) + \cos^2\theta_6\chi_1^{(g)}(p,t) - \sin 2\theta_6\left[\chi_6^{(g)}(p,t)\sin\phi_6 + \chi_5^{(g)}(p,t)\cos\phi_6\right] - \dot{\phi}_6\sin\theta_6, \quad (2.31)$$
$$\alpha_{-s}^{(g)}(p,t) = \sin^2\theta_5\chi_3^{(g)}(p,t) + \cos^2\theta_5\chi_2^{(g)}(p,t) - \sin 2\theta_5\left[\chi_8^{(g)}(p,t)\sin\phi_5 + \chi_7^{(g)}(p,t)\cos\phi_5\right] - \dot{\phi}_5\sin\theta_5,$$
$$\beta_{+s}^{(g)}(p,t) = \sin^2\theta_5\chi_2^{(g)}(p,t) + \cos^2\theta_5\chi_3^{(g)}(p,t) + \sin 2\theta_5\left[\chi_8^{(g)}(p,t)\sin\phi_5 + \chi_7^{(g)}(p,t)\cos\phi_5\right] - \dot{\phi}_5\sin\theta_5,$$
$$\beta_{-s}^{(g)}(p,t) = \sin^2\theta_6\chi_4^{(g)}(p,t) + \cos^2\theta_6\chi_1^{(g)}(p,t) + \sin 2\theta_6\left[\chi_6^{(g)}(p,t)\sin\phi_6 + \chi_5^{(g)}(p,t)\cos\phi_6\right] - \dot{\phi}_6\sin\theta_6,$$

where $\chi_m^{(g)}(p,t), (m=1,2,\ldots,8)$ can be found from Appendix Appendix B.. It is clearly seen from (2.28,2.30) that for each mode $\vec{p}$ in the p space, $\hat{H}_0(\vec{p},t)$ differs from $\hat{I}_V(\vec{p})$ only by a multiplying c-number factor depending on the time t and $p = |\vec{p}|$. $\hat{I}_V$ is time-independent and can be regarded in the discrete notation as a sum of terms each of which has the form of the Hamiltonian for a simple Fermi oscillator of frequency 1. The solutions to the oscillator eigenvalue problem for $\vec{p}_1, \vec{p}_2, \ldots$ modes may be characterized by integers $n_1, n_2, \ldots$ $(n_1, n_2, \ldots = 0, 1)$. The ground state of $\hat{I}_V(\vec{p})$ is denoted by $|0\rangle$ and satisfies

$$\hat{b}_s(\vec{p}_m)|0\rangle \equiv \hat{b}_{ms}|0\rangle = 0, \quad (2.32)$$
$$\hat{d}_s(\vec{p}_m)|0\rangle \equiv \hat{d}_{ms}|0\rangle = 0,$$

By making use of the ground state $|0\rangle$ and the raising operators $\hat{b}_s^+(\vec{p}_m) \equiv \hat{b}_{ms}^+, \hat{d}_s^+(\vec{p}_m) \equiv \hat{d}_{ms}^+$, we obtain the N-particle excited eigenstate $|N\rangle$ of $\hat{I}_V$ (with the particle number operators being defined to be $\hat{n}_{mbs} = \hat{b}_{ms}^+\hat{b}_{ms}$, $\hat{n}_{mds} = \hat{d}_{ms}^+\hat{d}_{ms}$, $\hat{N}_{bs} = \sum_m \hat{n}_{mbs}$, $\hat{N}_{ds} = \sum_m \hat{n}_{mds}$):

$$\begin{aligned} |N_{bs}\rangle &= |n_{1bs}, n_{2bs}, \ldots (n_{1bs} + n_{2bs} + \ldots = N_{bs})\rangle \quad (2.33)\\ &= \prod_m \left[\hat{b}_s^+(\vec{p}_m)\right]^{n_{mbs}} |0\rangle,\\ |N_{ds}\rangle &= |n_{1ds}, n_{2ds}, \ldots (n_{1ds} + n_{2ds} + \ldots = N_{ds})\rangle\\ &= \prod_m \left[\hat{d}_s^+(\vec{p}_m)\right]^{n_{mbs}} |0\rangle,\\ &\qquad (n_{mbs}, n_{mds} = 0, 1), \end{aligned}$$

which satisfies

$$\hat{I}_V |N_{bs}, N_{ds}\rangle_{I_V} = (N_{b+s} + N_{b-s} + N_{d+s} + N_{d-s}) |N_{bs}, N_{ds}\rangle_{I_V}; \quad (2.34)$$

for convenience, we define $|N_{bs}, N_{ds}\rangle_{I_V} \equiv |N_{b+s}\rangle |N_{b-s}\rangle |N_{d+s}\rangle |N_{d-s}\rangle$. According to (A9), the solutions $|N_{bs}, N_{ds}, t\rangle_{S0}$ of the time-dependent functional Schrödinger equation (associated with $\hat{H}_0(t)$) are

$$|N_{bs}, N_{ds}, t\rangle_{S0} = \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{bs}(t)\right] \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{ds}(t)\right] \times |N_{bs}, N_{ds}\rangle_{I_V}, \quad (2.35)$$

$$\begin{aligned} \vartheta_{bs} &= -\int_{t_0}^{t} \langle N_{bs}| \hat{H}_0(t') |N_{bs}\rangle \mathrm{d}t'\\ &= -\int_{t_0}^{t} \langle N_{bs}| \hat{U}^+(t') \hat{H}_0(t') \hat{U}(t')\\ &\quad -\mathrm{i}\hat{U}^+(t') \frac{\partial \hat{U}(t')}{\partial t} |N_{bs}\rangle \mathrm{d}t'\\ &= -\sum_m n_{mbs} \int_{t_0}^{t} \left[\alpha_s^{(d)}(p_m, t') + \alpha_s^{(g)}(p_m, t')\right] \mathrm{d}t', \end{aligned}$$

$$\begin{aligned} \vartheta_{ds} &= -\int_{t_0}^{t} \langle N_{ds}| \hat{H}_0(t') |N_{ds}\rangle \mathrm{d}t'\\ &= -\int_{t_0}^{t} \langle N_{ds}| \hat{U}^+(t') \hat{H}_0(t') \hat{U}(t')\\ &\quad -\mathrm{i}\hat{U}^+(t') \frac{\partial \hat{U}(t')}{\partial t} |N_{ds}\rangle \mathrm{d}t'\\ &= -\sum_m n_{mds} \int_{t_0}^{t} \left[\beta_s^{(d)}(p_m, t') + \beta_s^{(g)}(p_m, t')\right] \mathrm{d}t', \end{aligned}$$

in which $\vartheta_{bs}(t), \vartheta_{ds}(t)$ are the total phases, including the dynamical phases and geometrical phases. From (A6) in Appendix Appendix B. and (2.35), the particular exact solutions of the time-dependent Schrödinger equation (2.20) [associated with $\hat{H}(t)$] can be found to be

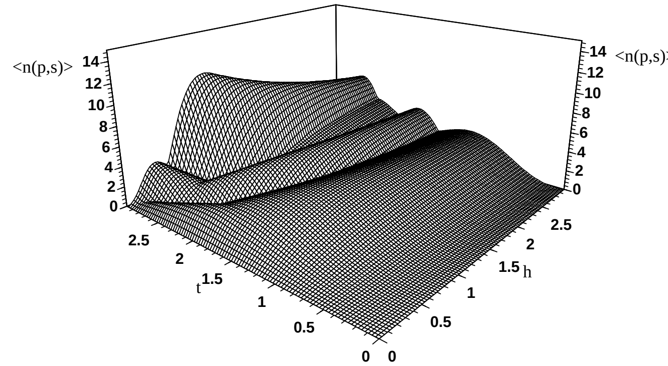

Fig. 1. The time and h (Hubble parameter) dependence of the expectation value $\langle n(\vec{p}, s)\rangle$ of the particle number $\hat{n}(\vec{p}, s) \equiv \hat{n}_{b+s} + \hat{n}_{b-s} + \hat{n}_{d+s} + \hat{n}_{d-s}$ for the state $|\Psi_{0,0}(t)\rangle_s$. For the purpose of illustration, $a(t)$ is taken to be of the form $a(t) = \mathrm{e}^{ht}$, $h = constant$, the particle mass $m = 1$, and the particle momentum $p_x = p_y = p_z = 1$

$$\begin{aligned} |\Psi_{N_{bs},N_{ds}}(t)\rangle_S &= \hat{V}(t) |N_{bs}, N_{ds}, t\rangle_{S0} \quad (2.36)\\ &= \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{bs}(t)\right] \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{ds}(t)\right] \hat{V}(t) |N_{bs}, N_{ds}\rangle_{I_V}\\ &= \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{bs}(t)\right] \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{ds}(t)\right] |N_{bs}, N_{ds}; t\rangle_I \end{aligned}$$

where $|N_{bs}, N_{ds}, t\rangle_I$ are the eigenstates of the invariant $\hat{I}(t)$ with particle number $(N_{b+s} + N_{b-s} + N_{d+s} + N_{d-s})$. The particular exact solutions in (2.36) for all possible n_{bs} and n_{ds} constitute a complete set; this means that the general exact solution of the time-dependent Schrödinger equation (2.20) is a superposition of the particular solutions in (2.36):

$$\begin{aligned} |\Psi(t)\rangle_S &= \sum_{N_{bs}N_{ds}} C_{N_{bs}N_{ds}} \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{bs}(t)\right]\\ &\quad \times \exp\left[\mathrm{i}\sum_{\pm s} \vartheta_{ds}(t)\right] |N_{bs}, N_{ds}, t\rangle_I \quad (2.37)\\ C_{N_{bs}N_{ds}} &= \langle N_{bs}, N_{ds}; 0|\Psi(0).\rangle_S \end{aligned}$$

It is worthwhile to point out that the auxiliary equations obtained in the present paper for the Dirac field are much more complicated than that for a scalar field in [11] and can only be solved numerically. As an illustration, we calculate the time dependence of the expectation value, denoted by $\langle \hat{n}(\vec{p}, s)\rangle_t$, of the particle number $\hat{n}(\vec{p}, s) \equiv \hat{n}_{b+s} + \hat{n}_{b-s} + \hat{n}_{d+s} + \hat{n}_{d-s}$ for the state $|\Psi_{0,0}(t)\rangle_s$, which is the ground state at $t = t_0$ corresponding to $\langle \hat{n}(\vec{p}, s)\rangle_{t_0} = 0$. Thus, the $\langle \hat{n}(\vec{p}, s)\rangle_t$ represents the particle creation. The result of the calculation is shown in Fig. 1.

3 The case in which there is an interaction between the Dirac field and a scalar field

In this section, we study the way of extending the formulation to treat the case in which there is an interaction between the Dirac field and a scalar field. As in the time-independent quantum field theory, we first try to find the solutions for the free Dirac field with external Dirac spinor sources $\zeta(\vec{x},t)$; the solutions can then be used to obtain the solutions for the system with an interaction between the Dirac field and a scalar field.

The classical Lagrangian density L_ζ for the free Dirac field with external Dirac spinor sources $\zeta(\vec{x},t)$ and $\bar{\zeta}(\vec{x},t)$ is

$$\begin{aligned}L(x) = {} & [-g(t)]^{\frac{1}{2}}\left\{\bar{\psi}(\vec{x})\left[\mathrm{i}\gamma_0\left(\frac{\partial}{\partial t}+u(t)\right)\right.\right.\\ & \left.\left.+\mathrm{i}a^{-1}(t)\nabla_i - m\right]\psi(\vec{x})\right\} \qquad (3.1)\\ & +[-g(t)]^{\frac{1}{2}}\left[\bar{\zeta}(\vec{x},t)\psi(\vec{x})+\bar{\psi}(\vec{x})\zeta(\vec{x},t)\right],\end{aligned}$$

where the chosen external Dirac spinor sources $\zeta(\vec{x},t)$ and $\bar{\zeta}(\vec{x},t)$ can be expressed as

$$\begin{aligned}\zeta(\vec{x},t) = {} & \sum_{\pm s}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E_p}}\left[l_s(\vec{p},t)\zeta_s(\vec{p})u_s(\vec{p})\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}}\right.\\ & \left.+l_s^*(\vec{p},t)\zeta_s^+(\vec{p})v_s(\vec{p})\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}}\right], \qquad (3.2)\\ \bar{\zeta}(\vec{x},t) = {} & \sum_{\pm s}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E_p}}\left[l_s^*(\vec{p},t)\zeta_s^+(\vec{p})\bar{u}_s(\vec{p})\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}}\right.\\ & \left.+l_s(\vec{p},t)\zeta_s(\vec{p})\bar{v}_s(\vec{p})\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}}\right],\end{aligned}$$

in which $l_s(\vec{p},t)$ are time-dependent c numbers and $\zeta_s(\vec{p})$ are Granssmann numbers. The quantum Hamiltonian $\hat{H}_\zeta(t)$ for this free Dirac field is

$$\begin{aligned}H_\zeta(t) = {} & \int\mathrm{d}^3\vec{p}\{\sum_{\pm s}[E_b\hat{b}_s^+\hat{b}_s - E_d\hat{d}_s\hat{d}_s^+ + \hat{b}_s^+\zeta_s(\vec{p},t)\\ & +\zeta_s^+(\vec{p},t)\hat{b}_s+\zeta_s(\vec{p},t)\hat{d}_s^+ + \hat{d}_s\zeta_s^+(\vec{p},t)]\\ & +\lambda_1(t)\hat{b}_{+s}^+\hat{d}_{+s}^+ + \lambda_2(t)\hat{b}_{-s}^+\hat{d}_{-s}^+ + \lambda_3(t)\hat{b}_{+s}^+\hat{d}_{-s}^+\\ & +\lambda_4(t)\hat{b}_{-s}^+\hat{d}_{+s}^+ + \lambda_1^*(t)\hat{d}_{+s}\hat{b}_{+s} + \lambda_2^*(t)\hat{d}_{-s}\hat{b}_{-s}\\ & +\lambda_3^*(t)\hat{d}_{-s}\hat{b}_{+s} + \lambda_4^*(t)\hat{d}_{+s}\hat{b}_{-s}\}, \qquad (3.3)\end{aligned}$$

in which $\zeta_s(\vec{p},t) = l_s'(\vec{p},t)\zeta_s(\vec{p}), l_s'(\vec{p},t) = a^3(t)l_s(\vec{p},t)$, and $E_b(t), E_d(t), \lambda_m(t)$ are defined in (2.19). The corresponding invariant $\hat{I}_\zeta(t)$ and the unitary transformation $\hat{V}_\zeta(t)$ are

$$\begin{aligned}\hat{I}_\zeta(t) = {} & \sum_{\pm s}\int\mathrm{d}^3\vec{p}\left[\hat{B}_{s\zeta}^+(\vec{p},t)\hat{B}_{s\zeta}(\vec{p},t)\right.\\ & \left.-\hat{D}_{s\zeta}(\vec{p},t)\hat{D}_{s\zeta}^+(\vec{p},t)\right], \qquad (3.4)\\ \hat{V}_\zeta(t) = {} & \hat{Q}(t)\hat{V}(t),\end{aligned}$$

where

$$\begin{aligned}\hat{B}_{+s}(\vec{p},t) = {} & \left[\cos\theta_1\cos\theta_2\cos\theta_6\right.\\ & \left.+\sin\theta_1\sin\theta_3\sin\theta_6\mathrm{e}^{-\mathrm{i}(\phi_1+\phi_3+\phi_6)}\right] \qquad (3.5)\\ & \times\left[\hat{b}_{+s}+\mu_1\mathrm{e}^{-\mathrm{i}v_1}\zeta_{+s}^+ + \mu_9\mathrm{e}^{-\mathrm{i}v_9}\zeta_{+s}\right.\\ & \left.+\mu_3\mathrm{e}^{-\mathrm{i}v_3}\zeta_{-s}^+ + \mu_{11}\mathrm{e}^{-\mathrm{i}v_{11}}\zeta_{-s}\right]\\ & +\left[\cos\theta_1\sin\theta_2\sin\theta_5\mathrm{e}^{\mathrm{i}(\phi_2+\phi_5)}\right.\\ & \left.-\sin\theta_1\cos\theta_3\cos\theta_5\mathrm{e}^{-\mathrm{i}\phi_1}\right]\\ & \times\left[\hat{b}_{-s}+\mu_2\mathrm{e}^{-\mathrm{i}v_2}\zeta_{-s}^+ + \mu_{10}\mathrm{e}^{-\mathrm{i}v_{10}}\zeta_{-s}\right.\\ & \left.+\mu_4\mathrm{e}^{-\mathrm{i}v_4}\zeta_{+s}^+ + \mu_{12}\mathrm{e}^{-\mathrm{i}v_{12}}\zeta_{+s}\right]\\ & +\left[\cos\theta_1\sin\theta_2\cos\theta_5\mathrm{e}^{\mathrm{i}\phi_2}\right.\\ & \left.+\sin\theta_1\cos\theta_3\sin\theta_5\mathrm{e}^{-\mathrm{i}(\phi_1+\phi_5)}\right]\\ & \times\left[\hat{d}_{+s}^+ + \mu_5\mathrm{e}^{\mathrm{i}v_5}\zeta_{+s}^+ + \mu_{13}\mathrm{e}^{\mathrm{i}v_{13}}\zeta_{+s}\right.\\ & \left.+\mu_7\mathrm{e}^{\mathrm{i}v_7}\zeta_{-s}^+ + \mu_{15}\mathrm{e}^{\mathrm{i}v_{15}}\zeta_{-s}\right]\\ & -\left[\cos\theta_1\cos\theta_3\sin\theta_6\mathrm{e}^{-\mathrm{i}\phi_6}\right.\\ & \left.-\sin\theta_1\sin\theta_3\cos\theta_6\mathrm{e}^{-\mathrm{i}(\phi_1+\phi_3)}\right]\\ & \times\left[\hat{d}_{-s}^+ + \mu_6\mathrm{e}^{\mathrm{i}v_6}\zeta_{-s}^+ + \mu_{14}\mathrm{e}^{\mathrm{i}v_{14}}\zeta_{-s}\right.\\ & \left.+\mu_8\mathrm{e}^{\mathrm{i}v_8}\zeta_{+s}^+ + \mu_{16}\mathrm{e}^{\mathrm{i}v_{16}}\zeta_{+s}\right],\end{aligned}$$

$$\begin{aligned}\hat{B}_{-s}(\vec{p},t) = {} & -\left[\cos\theta_1\sin\theta_3\sin\theta_6\mathrm{e}^{-\mathrm{i}(\phi_3+\phi_6)}\right.\\ & \left.-\sin\theta_1\cos\theta_2\cos\theta_6\mathrm{e}^{\mathrm{i}\phi_1}\right]\\ & \times\left[\hat{b}_{+s}+\mu_1\mathrm{e}^{-\mathrm{i}v_1}\zeta_{+s}^+ + \mu_9\mathrm{e}^{-\mathrm{i}v_9}\zeta_{+s}\right.\\ & \left.+\mu_3\mathrm{e}^{-\mathrm{i}v_3}\zeta_{-s}^+ + \mu_{11}\mathrm{e}^{-\mathrm{i}v_{11}}\zeta_{-s}\right]\\ & +\left[\cos\theta_1\cos\theta_3\cos\theta_5\right.\\ & \left.+\sin\theta_1\cos\theta_2\cos\theta_5\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2+\phi_5)}\right]\\ & \times\left[\hat{b}_{-s}+\mu_2\mathrm{e}^{-\mathrm{i}v_2}\zeta_{-s}^+ + \mu_{10}\mathrm{e}^{-\mathrm{i}v_{10}}\zeta_{-s}\right.\end{aligned}$$

$$
\begin{aligned}
&+\mu_4 \mathrm{e}^{-\mathrm{i}v_4} \zeta^+_{+s} + \mu_{12} \mathrm{e}^{-\mathrm{i}v_{12}} \zeta_{+s} \Big] \\
&-\Big[\cos\theta_1 \cos\theta_3 \sin\theta_5 \mathrm{e}^{-\mathrm{i}\phi_5} \\
&- \sin\theta_1 \sin\theta_2 \cos\theta_5 \mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)} \Big] \\
&\times \Big[\hat{d}^+_{+s} + \mu_5 \mathrm{e}^{\mathrm{i}v_5} \zeta^+_{+s} + \mu_{13} \mathrm{e}^{\mathrm{i}v_{13}} \zeta_{+s} \\
&+\mu_7 \mathrm{e}^{\mathrm{i}v_7} \zeta^+_{-s} + \mu_{15} \mathrm{e}^{\mathrm{i}v_{15}} \zeta_{-s} \Big] \\
&-\Big[\cos\theta_1 \sin\theta_3 \cos\theta_6 \mathrm{e}^{-\mathrm{i}\phi_3} \\
&+ \sin\theta_1 \cos\theta_2 \sin\theta_6 \mathrm{e}^{\mathrm{i}(\phi_1-\phi_6)} \Big] \\
&\times \Big[\hat{d}^+_{-s} + \mu_6 \mathrm{e}^{\mathrm{i}v_6} \zeta^+_{-s} + \mu_{14} \mathrm{e}^{\mathrm{i}v_{14}} \zeta_{-s} \\
&+\mu_8 \mathrm{e}^{\mathrm{i}v_8} \zeta^+_{+s} + \mu_{16} \mathrm{e}^{\mathrm{i}v_{16}} \zeta_{+s} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}^+_{-s}(\vec{p},t) = &\Big[\cos\theta_4 \cos\theta_3 \sin\theta_6 \mathrm{e}^{\mathrm{i}\phi_6} \\
&+ \sin\theta_4 \sin\theta_2 \cos\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_2+\phi_4)} \Big] \\
&\times \Big[\hat{b}_{+s} + \mu_1 \mathrm{e}^{-\mathrm{i}v_1} \zeta^+_{+s} + \mu_9 \mathrm{e}^{-\mathrm{i}v_9} \zeta_{+s} \\
&+\mu_3 \mathrm{e}^{-\mathrm{i}v_3} \zeta^+_{-s} + \mu_{11} \mathrm{e}^{-\mathrm{i}v_{11}} \zeta_{-s} \Big] \\
&+\Big[\cos\theta_4 \sin\theta_3 \cos\theta_5 \mathrm{e}^{\mathrm{i}\phi_3} \\
&- \sin\theta_4 \cos\theta_2 \sin\theta_5 \mathrm{e}^{-\mathrm{i}(\phi_4-\phi_5)} \Big] \\
&\times \Big[\hat{b}_{-s} + \mu_2 \mathrm{e}^{-\mathrm{i}v_2} \zeta^+_{-s} + \mu_{10} \mathrm{e}^{-\mathrm{i}v_{10}} \zeta_{-s} \\
&+\mu_4 \mathrm{e}^{-\mathrm{i}v_4} \zeta^+_{+s} + \mu_{12} \mathrm{e}^{-\mathrm{i}v_{12}} \zeta_{+s} \Big] \\
&-\Big[\cos\theta_4 \sin\theta_3 \sin\theta_5 \mathrm{e}^{\mathrm{i}(\phi_3-\phi_5)} \\
&+ \sin\theta_4 \cos\theta_2 \cos\theta_5 \mathrm{e}^{-\mathrm{i}\phi_4} \Big] \\
&\times \Big[\hat{d}^+_{+s} + \mu_5 \mathrm{e}^{\mathrm{i}v_5} \zeta^+_{+s} + \mu_{13} \mathrm{e}^{\mathrm{i}v_{13}} \zeta_{+s} \\
&+\mu_7 \mathrm{e}^{\mathrm{i}v_7} \zeta^+_{-s} + \mu_{15} \mathrm{e}^{\mathrm{i}v_{15}} \zeta_{-s} \Big] \\
&-\Big[\cos\theta_4 \cos\theta_3 \cos\theta_6 \\
&- \sin\theta_4 \sin\theta_2 \sin\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_2+\phi_4+\phi_6)} \Big] \\
&\times \Big[\hat{d}^+_{-s} + \mu_6 \mathrm{e}^{\mathrm{i}v_6} \zeta^+_{-s} + \mu_{14} \mathrm{e}^{\mathrm{i}v_{14}} \zeta_{-s} \\
&+\mu_8 \mathrm{e}^{\mathrm{i}v_8} \zeta^+_{+s} + \mu_{16} \mathrm{e}^{\mathrm{i}v_{16}} \zeta_{+s} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}^+_{+s}(\vec{p},t) = -&\Big[\cos\theta_4 \sin\theta_2 \cos\theta_6 \mathrm{e}^{-\mathrm{i}\phi_2} \\
&- \sin\theta_4 \cos\theta_3 \sin\theta_6 \mathrm{e}^{\mathrm{i}(\phi_4+\phi_6)} \Big] \\
&\times \Big[\hat{b}_{+s} + \mu_1 \mathrm{e}^{-\mathrm{i}v_1} \zeta^+_{+s} + \mu_9 \mathrm{e}^{-\mathrm{i}v_9} \zeta_{+s} \\
&+\mu_3 \mathrm{e}^{-\mathrm{i}v_3} \zeta^+_{-s} + \mu_{11} \mathrm{e}^{-\mathrm{i}v_{11}} \zeta_{-s} \Big] \\
&+\Big[\cos\theta_4 \cos\theta_2 \sin\theta_5 \mathrm{e}^{\mathrm{i}\phi_5} \\
&+ \sin\theta_4 \sin\theta_3 \cos\theta_5 \mathrm{e}^{\mathrm{i}(\phi_3+\phi_4)} \Big] \\
&\times \Big[\hat{b}_{-s} + \mu_2 \mathrm{e}^{-\mathrm{i}v_2} \zeta^+_{-s} + \mu_{10} \mathrm{e}^{-\mathrm{i}v_{10}} \zeta_{-s} \\
&+\mu_4 \mathrm{e}^{-\mathrm{i}v_4} \zeta^+_{+s} + \mu_{12} \mathrm{e}^{-\mathrm{i}v_{12}} \zeta_{+s} \Big] \\
&+\Big[\cos\theta_4 \cos\theta_2 \cos\theta_5 \\
&- \sin\theta_4 \sin\theta_3 \sin\theta_5 \mathrm{e}^{\mathrm{i}(\phi_3+\phi_4-\phi_5)} \Big] \\
&\times \Big[\hat{d}^+_{+s} + \mu_5 \mathrm{e}^{\mathrm{i}v_5} \zeta^+_{+s} + \mu_{13} \mathrm{e}^{\mathrm{i}v_{13}} \zeta_{+s} \\
&+\mu_7 \mathrm{e}^{\mathrm{i}v_7} \zeta^+_{-s} + \mu_{15} \mathrm{e}^{\mathrm{i}v_{15}} \zeta_{-s} \Big] \\
&+\Big[\cos\theta_4 \sin\theta_2 \sin\theta_6 \mathrm{e}^{-\mathrm{i}(\phi_2+\phi_6)} \\
&+ \sin\theta_4 \cos\theta_3 \cos\theta_6 \mathrm{e}^{\mathrm{i}\phi_4} \Big] \\
&\times \Big[\hat{d}^+_{-s} + \mu_6 \mathrm{e}^{\mathrm{i}v_6} \zeta^+_{-s} + \mu_{14} \mathrm{e}^{\mathrm{i}v_{14}} \zeta_{-s} \\
&+\mu_8 \mathrm{e}^{\mathrm{i}v_8} \zeta^+_{+s} + \mu_{16} \mathrm{e}^{\mathrm{i}v_{16}} \zeta_{+s} \Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{Q}(t) &= \hat{Q}_1(t)\, \hat{Q}_2(t)\, \hat{Q}_3(t)\, \hat{Q}_4(t), \qquad (3.6) \\
\hat{Q}_1(t) &= \exp \int \mathrm{d}^3\vec{p} a^3(t_0) \\
&\quad \times \Big\{ \Big[-\hat{b}^+_{+s} \left(\mu_1 \mathrm{e}^{-\mathrm{i}v_1} \zeta_{+s} + \mu_9 \mathrm{e}^{-\mathrm{i}v_9} \zeta^+_{+s} \right) \\
&\quad + \left(\mu_1 \mathrm{e}^{\mathrm{i}v_1} \zeta^+_{+s} + \mu_9 \mathrm{e}^{\mathrm{i}v_9} \zeta_{+s} \right) \hat{b}_{+s} \Big] \\
&\quad + \Big[-\hat{d}^+_{+s} \left(\mu_5 \mathrm{e}^{-\mathrm{i}v_5} \zeta_{+s} + \mu_{13} \mathrm{e}^{-\mathrm{i}v_{13}} \zeta^+_{+s} \right) \\
&\quad + \left(\mu_5 \mathrm{e}^{\mathrm{i}v_5} \zeta^+_{+s} + \mu_{13} \mathrm{e}^{\mathrm{i}v_{13}} \zeta_{+s} \right) \hat{d}_{+s} \Big] \Big\}, \\
\hat{Q}_2(t) &= \exp \int \mathrm{d}^3\vec{p} a^3(t_0)
\end{aligned}
$$

$$\times\left\{\left[-\hat{b}_{-s}^{+}\left(\mu_2 e^{-iv_2}\zeta_{-s}+\mu_{10}e^{-iv_{10}}\zeta_{-s}^{+}\right)+\left(\mu_2 e^{iv_2}\zeta_{-s}+\mu_{10}e^{iv_{10}}\zeta_{-s}^{+}\right)\hat{b}_{-s}\right]+\left[-\hat{d}_{-s}^{+}\left(\mu_6 e^{-iv_6}\zeta_{-s}+\mu_{14}e^{-iv_{14}}\zeta_{-s}^{+}\right)+\left(\mu_6 e^{iv_6}\zeta_{-s}^{+}+\mu_{14}e^{iv_{14}}\zeta_{-s}\right)\hat{d}_{-s}\right]\right\},$$

$$\hat{Q}_3(t)=\exp\int d^3\vec{p}a^3(t_0)\times\left\{\left[-\hat{b}_{+s}^{+}\left(\mu_3 e^{-iv_3}\zeta_{-s}+\mu_{11}e^{-iv_{11}}\zeta_{-s}^{+}\right)+\left(\mu_3 e^{iv_3}\zeta_{-s}^{+}+\mu_{11}e^{iv_{11}}\zeta_{-s}\right)\hat{b}_{+s}\right]+\left[-\hat{d}_{+s}^{+}\left(\mu_7 e^{-iv_7}\zeta_{-s}+\mu_{15}e^{-iv_{15}}\zeta_{-s}^{+}\right)+\left(\mu_7 e^{iv_7}\zeta_{-s}^{+}+\mu_{15}e^{iv_{15}}\zeta_{-s}\right)\hat{d}_{+s}\right]\right\},$$

$$\hat{Q}_4(t)=\exp\int d^3\vec{p}a^3(t_0)\times\left\{\left[-\hat{b}_{-s}^{+}\left(\mu_4 e^{-iv_4}\zeta_{+s}+\mu_{12}e^{-iv_{12}}\zeta_{+s}^{+}\right)+\left(\mu_4 e^{iv_4}\zeta_{+s}^{+}+\mu_{12}e^{iv_{12}}\zeta_{+s}\right)\hat{b}_{-s}\right]+\left[-\hat{d}_{-s}^{+}\left(\mu_8 e^{-iv_8}\zeta_{+s}+\mu_{16}e^{-iv_{16}}\zeta_{+s}^{+}\right)+\left(\mu_8 e^{iv_8}\zeta_{+s}^{+}+\mu_{16}e^{iv_{16}}\zeta_{+s}\right)\hat{d}_{-s}\right]\right\},$$

where $\mu_i, v_i, (i=1,2,\ldots,16)$ are the solutions of the auxiliary equations (see Appendix Appendix B.).

The unitary transformation $\hat{V}_\zeta(t)$ transforms $\hat{I}_\zeta(t)$ and $\hat{H}_\zeta(t)$ into $\hat{I}_{V\zeta}$ and $\hat{H}_{\zeta 0}(t)$,

$$\hat{I}_{V\zeta}=\hat{V}_\zeta^{+}(t)\hat{I}_\zeta(t)\hat{V}_\zeta(t) \tag{3.7}$$

$$\hat{H}_{\zeta 0}(t)=\hat{V}_\zeta^{+}(t)\hat{H}_\zeta(t)\hat{V}_\zeta(t)-\hat{V}_\zeta^{+}(t)\frac{\partial\hat{V}_\zeta(t)}{\partial t}=\hat{H}_0(t)+\int d^3\vec{p}\left[\sum_{\pm s}a_s\zeta_s^{+}\zeta_s+a_1\zeta_{+s}^{+}\zeta_{-s}+a_1^{*}\zeta_{-s}^{+}\zeta_{+s}+a_2\zeta_{+s}^{+}\zeta_{-s}^{+}+a_2^{*}\zeta_{-s}\zeta_{+s}\right],$$

where $\hat{H}_0(t)$ can be found from (2.27) and

$$\begin{aligned}a_{+s}=&E_b\left(\mu_1^2-\mu_9^2\right)+2\left(\dot{\mu}_1\mu_1-\dot{\mu}_9\mu_9\right)\\&-E_d\left(\mu_5^2-\mu_{13}^2\right)+2\left(\dot{\mu}_5\mu_5-\dot{\mu}_{13}\mu_{13}\right)\\&+E_b\left(\mu_4^2-\mu_{12}^2\right)+2\left(\dot{\mu}_4\mu_4-\dot{\mu}_{12}\mu_{12}\right)\\&-E_d\left(\mu_8^2-\mu_{16}^2\right)+2\left(\dot{\mu}_8\mu_8-\dot{\mu}_{16}\mu_{16}\right)\\&+2\left[l'_{+sr}\left(\mu_1\cos v_1-\mu_5\cos v_5\right)-l'_{+si}\left(\mu_1\sin v_1-\mu_5\sin v_5\right)\right]\\&+2\lambda_1\left[\mu_1\mu_{13}\cos(v_1+v_{13})-\mu_5\mu_9\cos(v_5+v_9)\right]\\&+2\lambda_2\left[\mu_4\mu_{16}\cos(v_4+v_{16})-\mu_8\mu_{12}\cos(v_8+v_{12})\right]\\&+2\mu_1\mu_{16}\left[\lambda_{3r}\cos(v_1+v_{16})-\lambda_{3i}\sin(v_1+v_{16})\right]\\&-2\mu_8\mu_9\left[\lambda_{3r}\cos(v_8+v_9)-\lambda_{3i}\sin(v_8+v_9)\right]\\&+2\mu_4\mu_{13}\left[\lambda_{4r}\cos(v_4+v_{13})-\lambda_{4i}\sin(v_4+v_{13})\right]\\&-2\mu_5\mu_{12}\left[\lambda_{4r}\cos(v_5+v_{12})-\lambda_{4i}\sin(v_5+v_{12})\right],\end{aligned} \tag{3.8}$$

$$\begin{aligned}a_{-s}=&E_b\left(\mu_3^2-\mu_{11}^2\right)+2\left(\dot{\mu}_3\mu_3-\dot{\mu}_{11}\mu_{11}\right)\\&-E_d\left(\mu_6^2-\mu_{14}^2\right)+2\left(\dot{\mu}_6\mu_6-\dot{\mu}_{14}\mu_{14}\right)\\&+E_b\left(\mu_2^2-\mu_{10}^2\right)+2\left(\dot{\mu}_2\mu_2-\dot{\mu}_{10}\mu_{10}\right)\\&-E_d\left(\mu_7^2-\mu_{15}^2\right)+2\left(\dot{\mu}_7\mu_7-\dot{\mu}_{15}\mu_{15}\right)\\&+2\left[l'_{-sr}\left(\mu_3\cos v_3-\mu_7\cos v_7\right)-l'_{-si}\left(\mu_3\sin v_3-\mu_7\sin v_7\right)\right]\\&+2\lambda_1\left[\mu_3\mu_{15}\cos(v_3+v_{15})-\mu_7\mu_{11}\cos(v_7+v_{11})\right]\\&+2\lambda_2\left[\mu_2\mu_{14}\cos(v_2+v_{14})-\mu_6\mu_{10}\cos(v_6+v_{10})\right]\\&+2\mu_3\mu_{14}\left[\lambda_{3r}\cos(v_3+v_{14})-\lambda_{3i}\sin(v_3+v_{14})\right]\\&-2\mu_6\mu_{11}\left[\lambda_{3r}\cos(v_6+v_{11})-\lambda_{3i}\sin(v_6+v_{11})\right]\\&+2\mu_2\mu_{15}\left[\lambda_{4r}\cos(v_2+v_{15})-\lambda_{4i}\sin(v_2+v_{15})\right]\\&-2\mu_7\mu_{10}\left[\lambda_{4r}\cos(v_7+v_{10})-\lambda_{4i}\sin(v_7+v_{10})\right],\end{aligned}$$

$$\begin{aligned}a_1=&E_b\left[\mu_1\mu_3 e^{i(v_1+v_3)}-\mu_9\mu_{11}e^{-i(v_9-v_{11})}+\mu_2\mu_4 e^{i(v_2-v_4)}-\mu_{10}\mu_{12}e^{-i(v_{10}-v_{12})}\right]\\&-E_d\left[\mu_5\mu_7 e^{i(v_5-v_7)}-\mu_{13}\mu_{15}e^{-i(v_{13}-v_{15})}+\mu_6\mu_8 e^{i(v_6-v_8)}-\mu_{16}\mu_{14}e^{-i(v_{16}-v_{14})}\right]\\&+l'_{-s}\mu_1 e^{iv_1}+l'_{+s}\mu_{10}e^{-iv_{10}}-l'_{-s}\mu_5 e^{iv_5}-l'_{+s}\mu_6 e^{-iv_6}\\&+2\lambda_1\left[\mu_1\mu_{15}\cos(v_1+v_{15})-\mu_5\mu_{11}\cos(v_5+v_{11})\right]\\&+2\lambda_2\left[\mu_4\mu_{14}\cos(v_4+v_{14})-\mu_{10}\mu_8\cos(v_{10}+v_8)\right]\\&+2\mu_1\mu_{14}\left[\lambda_{3r}\cos(v_1+v_{14})-\lambda_{3i}\sin(v_1+v_{14})\right]\\&-2\mu_{11}\mu_8\left[\lambda_{3r}\cos(v_{11}+v_8)-\lambda_{3i}\sin(v_{11}+v_8)\right]\\&+2\mu_4\mu_{15}\left[\lambda_{4r}\cos(v_4+v_{15})-\lambda_{4i}\sin(v_4+v_{15})\right]\\&-2\mu_5\mu_{10}\left[\lambda_{4r}\cos(v_5+v_{10})-\lambda_{4i}\sin(v_5+v_{10})\right],\end{aligned}$$

$$\begin{aligned}a_2=&E_b\left[\mu_1\mu_{11}e^{i(v_1+v_{11})}-\mu_9\mu_3 e^{-i(v_9-v_3)}+\mu_{10}\mu_4 e^{i(v_{10}-v_4)}-\mu_2\mu_{12}e^{i(v_2-v_{12})}\right]\\&-E_d\left[\mu_5\mu_{15}e^{i(v_5-v_{15})}-\mu_{13}\mu_7 e^{-i(v_{13}-v_7)}+\mu_{14}\mu_8 e^{-i(v_{14}-v_8)}-\mu_{16}\mu_6 e^{-i(v_{16}-v_6)}\right]\\&+l'_{+s}\mu_{11}e^{-iv_{11}}+l'_{-s}\mu_{13}e^{-iv_{13}}\\&-l'_{-s}\mu_{12}e^{iv_{12}}-l'_{+s}\mu_{15}e^{iv_{15}}\\&+2\lambda_1\left[\mu_1\mu_7\cos(v_1+v_7)-\mu_5\mu_3\cos(v_5+v_3)\right]\\&+2\lambda_2\left[\mu_4\mu_6\cos(v_4+v_6)-\mu_2\mu_8\cos(v_2+v_8)\right]\\&+2\mu_1\mu_6\left[\lambda_{3r}\cos(v_1+v_6)-\lambda_{3i}\sin(v_1+v_6)\right]\\&-2\mu_3\mu_8\left[\lambda_{3r}\cos(v_3+v_8)-\lambda_{3i}\sin(v_3+v_8)\right]\\&+2\mu_2\mu_{15}\left[\lambda_{4r}\cos(v_2+v_{15})-\lambda_{4i}\sin(v_2+v_{15})\right]\\&-2\mu_7\mu_{10}\left[\lambda_{4r}\cos(v_7+v_{10})-\lambda_{4i}\sin(v_7+v_{10})\right].\end{aligned}$$

Then, in the same way as in Sect. 2, the corresponding solutions of the time-dependent functional Schrödinger equation (associated with $\hat{H}_\zeta(t)$) can be found to be

$$\begin{aligned} |\Psi_{N;\zeta}(t)\rangle_S &\equiv |\Psi_{N_{bs}N_{ds};\zeta}(t)\rangle_S \\ &= \hat{V}_\zeta(t)\,|N_{bs},N_{ds};t\rangle_{\zeta S0} \qquad (3.9)\\ &= \exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{bs}(t)\right]\exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{ds}(t)\right] \\ &\quad\times\exp\left[\mathrm{i}\vartheta_\zeta(t)\right]\hat{V}_\zeta(t)\,|N_{bs},N_{ds}\rangle_{I_U} \\ &= \exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{bs}(t)\right]\exp\left[\mathrm{i}\sum_{\pm s}\vartheta_{ds}(t)\right] \\ &\quad\times\exp\left[\mathrm{i}\vartheta_\zeta(t)\right]|N_{bs},N_{ds};t\rangle_{I_\zeta}, \\ \Psi_{N;\zeta}[\psi,t] &= \langle\psi|\Psi_{N;\zeta}(t)\rangle_S \end{aligned}$$

where $|N_{bs},N_{ds};t\rangle_{I_\zeta}$ are the eigenstates of the invariant $\hat{I}_\zeta(t)$ in (3.7), which are calculated as in Sect. 2; $\vartheta_{bs}(t)$ and $\vartheta_{ds}(t)$ can be found from (2.35); and

$$\vartheta_\zeta(t) = \int \mathrm{d}^3\vec{p}\int_{t_0}^{t}\mathrm{d}t'\left[\sum_{\pm s} a_s\zeta_s^{+}\zeta_s + a_1\zeta_{+s}^{+}\zeta_{-s} + a_1^{*}\zeta_{-s}^{+}\zeta_{+s} + a_2\zeta_{+s}^{+}\zeta_{-s}^{+} + a_2^{*}\zeta_{-s}\zeta_{+s}\right] \qquad (3.10)$$

Now we proceed to find the solutions of the time-dependent Schrödinger equation for the system with an interaction between the Dirac field and a scalar field. The classical Lagrangian density $L_{\zeta J}^{f\lambda}$ can be written as

$$\begin{aligned} L_{\zeta J}^{f\lambda} &= L_\zeta + L_J + L_I, \qquad (3.11)\\ L_J &= \frac{1}{2}\left[-g(t)\right]^{\frac{1}{2}}\left\{\dot{\phi}^2 - a^{-1}(t)\left|\nabla\phi\right|^2 - \left[m+u'(t)\right]\phi^2 - \frac{\lambda}{4!}\phi^4\right\} + \left[-g(t)\right]^{\frac{1}{2}} J(\vec{x},t)\,\phi, \\ L_\zeta &= \left[-g(t)\right]^{\frac{1}{2}}\left\{\bar{\psi}\left[\mathrm{i}\gamma_0\left(\frac{\partial}{\partial t}+u(t)\right) + \mathrm{i}a^{-1}(t)\,\nabla_i - m\right]\psi\right\} \\ &\quad + \left[-g(t)\right]^{\frac{1}{2}}\left[\bar{\zeta}(\vec{x},t)\,\psi + \bar{\psi}\zeta(\vec{x},t)\right], \\ L_I &= -\left[-g(t)\right]^{\frac{1}{2}} f\bar{\psi}\phi\psi, \end{aligned}$$

where L_J can be found from [11]. The solutions $\Psi_{N;\zeta,J}^{f\lambda}[\psi,\phi;t]$ of the time-dependent Schrödinger equation associated with the Hamiltonian $\hat{H}_{\zeta,J}^{f\lambda}$ corresponding to $L_{\zeta,J}^{f\lambda}$ can be written as

$$\begin{aligned} \Psi_{N_1,N_2;\zeta,J}^{f\lambda}[\psi,\phi;t] &= \left\langle\psi,\phi|\Psi_{N_1,N_2;\zeta,J}^{f\lambda}(t)\right\rangle \qquad (3.12)\\ &= \int D\phi_0 D\psi_0\,\langle\psi,\phi|\,\hat{U}_{\zeta,J}^{f}(t,t_0)\,|\psi_0,\phi_0\rangle \\ &\quad\times\langle\psi_0,\phi_0|\Psi_{N_1;\zeta}(t_0),\Psi_{N_2;J}(t_0)\rangle, \end{aligned}$$

where $|\Psi_{N_1;\zeta}(t_0),\Psi_{N_2;J}(t_0)\rangle = |\Psi_{N_1;\zeta}(t_0)\rangle\,|\Psi_{N_2;J}(t_0)\rangle$ is the initial condition, and $|\Psi_{N_1;\zeta}(t_0)\rangle$ and $|\Psi_{N_2;J}(t_0)\rangle$ are defined in (3.9) and [11], respectively. $\hat{U}_{\zeta,J}^{f\lambda}(t,t_0)$ is the time-evolution operator for the system with the Hamiltonian $\hat{H}_{\zeta,J}^{f\lambda}$ corresponding to $L_{\zeta,J}^{f\lambda}$, and $\langle\psi,\phi|\,\hat{U}_{\zeta,J}^{f\lambda}(t,t_0)\,|\psi_0,\phi_0\rangle$ can be calculated by path-integral approach,

$$\begin{aligned} &\langle\psi,\phi|\,\hat{U}_{\zeta,J}^{f\lambda}(t,t_0)\,|\psi_0,\phi_0\rangle \qquad (3.13)\\ &= \int D\phi_1\ldots D\phi_n D\psi_1\ldots D\psi_n D\psi_0^{+}\ldots D\psi_n^{+} \\ &\times\exp\left[\mathrm{i}\int\mathrm{d}^3\vec{x}\int_{t_0}^{t}\mathrm{d}t' L_{\zeta,J}^{f\lambda}\left(\psi,\psi^{+},\phi,t'\right)\right] \\ &= \exp\left\{-\int\mathrm{d}^3\vec{x}\int_{t_0}^{t}\mathrm{d}t'\left[a^3(t')\,f\right]\right. \\ &\times\frac{\delta}{a^3(t')\,\delta\zeta^{+}(\vec{x},t')}\frac{\delta}{a^3(t')\,\delta J(\vec{x},t')}\frac{\delta}{a^3(t')\,\delta\zeta(\vec{x},t')} \\ &\left.-\mathrm{i}a^3(t)\frac{\lambda}{4!}\left[\frac{\delta}{a^3(t')\,\delta J(\vec{x},t')}\right]^4\right\}\langle\psi,\phi|\,\hat{U}_{\zeta,J}(t,t_0)\,|\psi_0,\phi_0\rangle \end{aligned}$$

where $\hat{U}_{\zeta,J}(t,t_0) = P\exp\left[-\mathrm{i}\int\hat{H}_{\zeta,J}(t')\,\mathrm{d}t'\right]$ is the time-evolution operator for the free Dirac field and scalar field with external sources. Using (3.12) and (3.13), we obtain

$$\begin{aligned} \Psi_{N;\zeta,J}^{f\lambda}[\psi,\phi;t] &= \exp\left\{-\int\mathrm{d}^3\vec{x}\int_{t_0}^{t}\mathrm{d}t'\left[a^3(t')\,f\right]\right. \\ &\times\frac{\delta}{a^3(t')\,\delta\zeta^{+}(x,t')}\frac{\delta}{a^3(t')\,\delta J(x,t')}\frac{\delta}{a^3(t')\,\delta\zeta(x,t')} \\ &\left.-\mathrm{i}a^3(t)\frac{\lambda}{4!}\left[\frac{\delta}{a^3(t')\,\delta J(\vec{x},t')}\right]^4\right\} \\ &\times\Psi_{N;J}[\phi;t]\,\Psi_{N;\zeta}[\psi;t], \qquad (3.14) \end{aligned}$$

where $\Psi_{N;\zeta}[\psi;t]$ can be found from (3.9) and $\Psi_{N;J}[\phi;t]$ can be found from [11]. (3.14) can be employed as a starting point for the perturbative calculation of $\Psi_{N;\zeta=0,J=0}^{f\lambda}[\psi,\phi;t]$ when the parameters f and λ are small.

4 Concluding remarks

In this paper:

1. We have obtained all the excited-state as well as the ground-state solutions of the time-dependent functional Schrödinger equation (2.20). These solutions are also the eigenfunctions of $\hat{I}(t)$ and constitute a complete set of solutions. It is this set that can be used to treat the case in which the Dirac field is coupled to a scalar field. Note that even when we choose $\hat{H}(t=t_0)$ as $\hat{I}_0$, the ground-state solution is the ground eigenstate of the Hamiltonian $\hat{H}(t)$ only at $t=t_0$, so that, in general, the term "ground-state" is without the meaning that it has in the case in which the Hamiltonian is time-independent.

2. It can be shown that $\hat{b}_s^+(\vec{p}), \hat{b}_s(\vec{p}), \hat{d}_s^+(\vec{p})$, and $\hat{d}_s(\vec{p})$ in (2.17) and the quantum Hamiltonian $\hat{H}(t)$ in (2.18) constitute a quasialgebra [19]. It is this algebra that makes it possible to find the unitary transformation $\hat{V}(t)$ in (2.26). For other Hamiltonians, it is also necessary to study whether or not there exists such a quasialgebra in order to obtain the corresponding unitary transformation.
3. It is worthwhile to point out that the invariant-related unitary transformation (2.25) is a special kind of the Bogoliubov transformation (the invariant-related Bogoliubov transformation) and can be used in the Schrödinger picture quantum field theory to obtain the solutions of (2.17) on the basis of the quantum-invariant theory and to construct the generalized particle number operators, as well as the generalized annihilation and creation operators that are all invariants, and can be employed to generate all solutions of the time-dependent functional Schrödinger equation (2.17) from the ground-state solution $|\Psi_{0,0}(t)\rangle_S$ of it [11,12]; it is thus especially useful and different from the Bogoliubov transformation used in [3,4].
4. With the help of (3.14), we can carry out a variety of perturbative calculations to any desired order. In these calculations, one is bound to encounter divergences and has to develop some regularization and renormalization procedures; these deserve further exploration.
5. From the invariant formulation, briefly presented in Appendix Appendix A., it is easy to see the fact that the time-evolution operator $\hat{U}(t)$ associated with the Schrö-
dinger equation (A3) can be obtained from the invariant-related unitary transformation $\hat{V}(t)$ and the time-evolution operator $\hat{U}_0(t)$ associated with the Schrödinger equation (A8), and that the latter is very easy to obtain. This fact applies, of course, to the case discussed in Sect. 2. This is to say that we can obtain the time-evolution operator associated with the functional Schrödinger equation (2.20) from the invariant-related unitary transformation $\hat{V}(t)$ in (2.26).

If the scalar field ϕ in the interaction term $\bar{\psi}\phi\psi$ in Sect. 3 is replaced by a mean field $\langle\phi\rangle$ (depending only on time), the case considered in Sect. 3 is reduced to a free Dirac field case, which can be solved exactly as in Sect. 2 to obtain the mean-field approximation $U_{mf}^f(t,t_0)$ of the time-evolution operator for the Dirac field interacting with a scalar field. Using the method in [11], we can also get the mean-field approximation $U_{mf}^\lambda(t,t_0)$ of the time-evolution operator for the scalar field if the self-interaction term $(\lambda/4!)\phi^4$ is replaced by its mean-field approximation as the authors did in [1]. The mean-field approximation of the time-evolution operator for the Dirac interacting with a scalar field is therefore obtained to be $U_{mf}^{f\lambda} = U_{mf}^f(t,t_0)\,U_{mf}^\lambda(t,t_0)$. It is easily seen that, with the help of $U_{mf}^{f\lambda}(t,t_0)$, one can carry out the same sort of calculation in the nonperturbative Hartree approximation for a quantum Dirac field interacting with a scalar field in FRW spacetimes, just as the authors did in [1] for a quantum scalar field with a self-interaction in a collisionless approximation. Work in this direction is under investigation.

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Appendix A. The invariant-related unitary transformation method

We briefly review the invariant-related unitary transformation method. Consider a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent. A Hermitian operator $\hat{I}(t)$ is called an invariant if it satisfies

$$\frac{\partial \hat{I}(t)}{\partial t} - \left(\frac{\mathrm{i}}{\hbar}\right)\left[\hat{I}(t), \hat{H}(t)\right] = 0. \tag{A1}$$

The eigenvalue equation of $\hat{I}(t)$ can be written as

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{A2}$$
$$\frac{\partial\lambda_n}{\partial t} = 0,$$

and the time-dependent Schrödinger equation for the system is

$$\mathrm{i}\hbar\frac{\partial|\Psi(t)\rangle_S}{\partial t} = \hat{H}(t)|\Psi(t)\rangle_S. \tag{A3}$$

According to the Lewis–Riesenfeld quantum-invariant theory [14], the particular solution $|\lambda_n, t\rangle_S$ of (A3) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[\mathrm{i}\varphi_n(t)]$; that is,

$$|\lambda_n, t\rangle_S = \exp[\mathrm{i}\varphi_n(t)]|\lambda_n, t\rangle. \tag{A4}$$

Then the general solution of the Schrödinger equation (A3) can be shown to be

$$|\Psi(t)\rangle_S = \sum_n C_n \exp[\mathrm{i}\varphi_n(t)]|\lambda_n, t\rangle \tag{A5}$$

where $\varphi_n(t) = \int_0^t \langle\lambda_n, t'|\,\mathrm{i}\partial/\partial t' - \hat{H}(t')/\hbar\,|\lambda_n, t'\rangle\,\mathrm{d}t'$, $C_n = \langle\lambda_n, 0|\Psi(0)\rangle_S$, $|\lambda_n, t\rangle_S$ $(n = 1, 2, \ldots)$ are said to form a complete set of the solutions of (A3). Note that generally, $\hat{I}(t)$ is not unique, and the complete set changes as the choice of $\hat{I}(t)$ changes.

In [15], we generalized the Lewis–Riesenfeld-invariant theory and established the facts: (i) the formal solution of (A1) is $\hat{I}(t) = \hat{U}(t)\hat{I}(0)\hat{U}^+(t)$, where $\hat{U}(t) = P\exp\left[(-\mathrm{i}/\hbar)\int_0^t \hat{H}(t')\,\mathrm{d}t'\right]$ is the time-evolution operator for the system, and $I(0)$ can be arbitrarily chosen so that $\hat{I}(t)$ may be Hermitian or non-Hermitian; (ii) there are two basic invariants, $\hat{x}(t) = \hat{U}(t)\hat{x}\hat{U}^+(t)$ *and* $\hat{p}(t) = \hat{U}(t)\hat{p}\hat{U}^+(t)$; any invariant $\hat{I}(t) = \hat{U}(t)\hat{I}(0)\hat{U}^+(t)$ can be expressed as a power series in $\hat{x}(t)$ and $\hat{p}(t)$ as long as $\hat{I}(0)$ can be expressed as a power series in $\hat{x}$ and $\hat{p}$; (iii) in some cases, a chosen non-Hermitian invariant can act as a solution generator, with which one can generate a complete set of solutions of the time-dependent Schrödinger equation (A3) from one solution of it; (iv) the concept can be generalized so that a complete set of invariants can be found and an invariant formulation set up for the study of more-than-one-dimensional time-dependent systems (including infinite-dimensional quantum systems or quantum fields [10,11,16]).

Now we begin to briefly review the invariant-related unitary transformation method on the basis of the generalized invariant theory. In some cases of physical interest, it is possible to construct a time-dependent unitary transformation $\hat{V}(t)$ for a chosen invariant $\hat{I}(t)$ such that (i) $\hat{I}_0 \equiv \hat{V}^+(t)\,\hat{I}(t)\,\hat{V}(t)$ is a time-independent operator with

$$\hat{I}_0\left|\lambda_n\right\rangle = \lambda_n\left|\lambda_n\right\rangle, \qquad \left|\lambda_n\right\rangle = \hat{V}^{-1}(t)\left|\lambda_n, t\right\rangle, \tag{A6}$$

the eigenvalue λ_n being the same as that in (A2), and (ii) the Hamiltonian $\hat{H}(t)$ is changed into $\hat{H}_0(t)$:

$$\hat{H}_0(t) = \hat{V}^+(t)\,\hat{H}(t)\,\hat{V}(t) - \mathrm{i}\hbar\hat{V}^+(t)\,\frac{\partial\hat{V}(t)}{\partial t}. \tag{A7}$$

This unitary transformation is easily shown to guarantee that the particular solution $\left|\lambda_n, t\right\rangle_{S0}$ of the time-dependent Schrödinger equation [associated with $\hat{H}_0(t)$],

$$\mathrm{i}\hbar\frac{\partial\left|\lambda_n, t\right\rangle_{S0}}{\partial t} = \hat{H}_0(t)\left|\lambda_n, t\right\rangle_{S0}, \tag{A8}$$

is different from the eigenfunction $\left|\lambda_n\right\rangle$ of $\hat{I}_0$ only by the same phase factor $\exp[\mathrm{i}\varphi_n(t)]$ as that in (A4):

$$\left|\lambda_n, t\right\rangle_{S0} = \exp\left[\mathrm{i}\varphi_n(t)\right]\left|\lambda_n\right\rangle. \tag{A9}$$

Substitution of $\left|\lambda_n, t\right\rangle_{S0}$ in (A9) into (A8) yields

$$-\hbar\dot{\varphi}_n\left|\lambda_n\right\rangle = \hat{H}_0(t)\left|\lambda_n\right\rangle, \tag{A10}$$

which means that $\hat{H}_0(t)$ differs from $\hat{I}_0$ only by a multiplying c-number factor, depending on the time t. Thus, one is led to the conclusion that if the unitary transformation $\hat{V}(t)$, $\hat{I}_0, \hat{H}_0$, and the eigenfunction $\left|\lambda_n\right\rangle$ of $\hat{I}_0$ have been found, the problem of solving the complicated time-dependent Schrödinger equation (A3) reduces to that of solving the much simplified equation (A8) since it can be seen from (A9) that the solution of (A8) can be easily obtained by the calculation of the phase from (A10).

It is worthwhile to emphasize that (i) the term "a chosen Invariant" used above implies that the choice of the invariant $\hat{I}(t)$ is not unique, and it is usually appropriate to choose $\hat{I}(t) = \hat{U}\hat{I}(0)\,\hat{U}^+$, as the system is initially in an eigenstate of an operator $I(0)$ [$I(0)$ may be $\hat{H}(t=0)$]; and (ii) one chosen invariant $\hat{I}(t)$ leads to one definite complete set of the solutions $\left|\lambda_n, t\right\rangle$ of (A3), regardless of the fact that the unitary transformation $\hat{V}$, which is required to make $\hat{V}^+\hat{I}\hat{V}$ time-independent, is only determined up to a time-independent unitary transformation.

Appendix B. The auxiliary equations

The auxiliary equations for θ_m and ϕ_m $(m = 1\ldots6)$ in Sect. 2 are as follows:

$$\begin{aligned}\dot{\theta}_1 = {}& \frac{1}{2\left(\cos^2\theta_2 - \sin^2\theta_3\right)}\{\sin 2\theta_3\cos\theta_1\cos\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_3+\phi_4) + \lambda_{3i}\cos(\phi_3+\phi_4)\right] \\ & + \sin 2\theta_3\sin\theta_1\sin\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_3+\phi_4) - \lambda_{3i}\cos(\phi_3+\phi_4)\right] \\ & - \sin 2\theta_2\sin\theta_1\sin\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_2+\phi_4) - \lambda_{3i}\cos(\phi_2+\phi_4)\right] \\ & - \sin 2\theta_2\cos\theta_1\cos\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_1+\phi_2) + \lambda_{3i}\cos(\phi_1+\phi_2)\right] \\ & + \lambda_1\sin 2\theta_2\cos\theta_1\sin\theta_4\sin(\phi_1+\phi_2+\phi_4) \\ & + \lambda_1\sin 2\theta_3\sin\theta_1\cos\theta_4\sin\phi_3 \\ & - \lambda_1\sin 2\theta_3\cos\theta_1\sin\theta_4\sin(\phi_1+\phi_3+\phi_4) \\ & - \lambda_1\sin 2\theta_2\sin\theta_1\cos\theta_4\sin\phi_2\end{aligned} \tag{B1}$$

$$\begin{aligned}\dot{\theta}_2 = {}& \cos\theta_4\sin\theta_1\left[\lambda_{3r}\sin(\phi_1+\phi_2) + \lambda_{3i}\cos(\phi_1+\phi_2)\right] \\ & - \lambda_1\cos\theta_4\cos\theta_1\sin\phi_2 \\ & - \cos\theta_1\sin\theta_4\left[\lambda_{3r}\sin(\phi_2+\phi_4) + \lambda_{3i}\cos(\phi_2+\phi_4)\right] \\ & - \lambda_1\sin\theta_4\sin\theta_1\sin(\phi_1+\phi_2+\phi_4)\end{aligned} \tag{B2}$$

$$\begin{aligned}\dot{\theta}_3 = {}& \cos\theta_1\sin\theta_4\left[\lambda_{3r}\sin(\phi_3+\phi_4) - \lambda_{3i}\cos(\phi_3+\phi_4)\right] \\ & + \lambda_1\cos\theta_4\cos\theta_1\sin\phi_3 \\ & - \cos\theta_4\sin\theta_1\left[\lambda_{3r}\sin(\phi_1+\phi_3) + \lambda_{3i}\cos(\phi_1+\phi_3)\right] \\ & + \lambda_1\sin\theta_4\sin\theta_1\sin(\phi_1+\phi_3+\phi_4)\end{aligned} \tag{B3}$$

$$\begin{aligned}\dot{\theta}_4 = {}& \frac{1}{2\left(\cos^2\theta_2 - \sin^2\theta_3\right)}\{\sin 2\theta_2\cos\theta_1\cos\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_2+\phi_4) - \lambda_{3i}\cos(\phi_2+\phi_4)\right] \\ & + \sin 2\theta_2\sin\theta_1\sin\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_1+\phi_2) + \lambda_{3i}\cos(\phi_1+\phi_2)\right] \\ & - \sin 2\theta_3\sin\theta_1\sin\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_1+\phi_3) + \lambda_{3i}\cos(\phi_1+\phi_3)\right] \\ & - \sin 2\theta_3\cos\theta_1\cos\theta_4 \\ & \times\left[\lambda_{3r}\sin(\phi_3+\phi_4) - \lambda_{3i}\cos(\phi_3+\phi_4)\right] \\ & + \lambda_1\sin 2\theta_2\cos\theta_4\sin\theta_1\sin(\phi_1+\phi_2+\phi_4) \\ & + \lambda_1\sin 2\theta_3\sin\theta_4\cos\theta_1\sin\phi_3 \\ & - \lambda_1\sin 2\theta_3\cos\theta_4\sin\theta_1\sin(\phi_1+\phi_3+\phi_4) \\ & - \lambda_1\sin 2\theta_2\sin\theta_4\cos\theta_1\sin\phi_2\end{aligned} \tag{B4}$$

$$\begin{aligned}\dot{\theta}_5 = {}& \cos\phi_5\left[\chi_8^{(d)}(p,t) + \chi_8^{(g)}(p,t)\right] \\ & - \sin\phi_5\left[\chi_7^{(d)}(p,t) + \chi_7^{(g)}(p,t)\right]\end{aligned} \tag{B5}$$

$$\begin{aligned}\dot{\theta}_6 = {}& \cos\phi_6\left[\chi_6^{(d)}(p,t) + \chi_6^{(g)}(p,t)\right] \\ & - \sin\phi_6\left[\chi_5^{(d)}(p,t) + \chi_5^{(g)}(p,t)\right]\end{aligned} \tag{B6}$$

$$\dot{\phi}_1 = \frac{\csc 2\theta_1}{2\left(\cos^2\theta_2 - \sin^2\theta_3\right)}\{\sin 2\theta_3 \cos\theta_1 \cos\theta_4 \times [\lambda_{3r}\cos(\phi_3+\phi_4) - \lambda_{3i}\sin(\phi_3+\phi_4)] + \sin 2\theta_3 \sin\theta_1 \sin\theta_4 \times [\lambda_{3r}\cos(\phi_3+\phi_4) + \lambda_{3i}\sin(\phi_3+\phi_4)] - \sin 2\theta_2 \sin\theta_1 \sin\theta_4 \times [\lambda_{3r}\cos(\phi_2+\phi_4) + \lambda_{3i}\sin(\phi_2+\phi_4)] - \sin 2\theta_2 \cos\theta_1 \cos\theta_4 \times [\lambda_{3r}\cos(\phi_1+\phi_2) - \lambda_{3i}\sin(\phi_1+\phi_2)] + \lambda_1 \sin 2\theta_2 \cos\theta_1 \sin\theta_4 \cos(\phi_1+\phi_2+\phi_4) + \lambda_1 \sin 2\theta_3 \sin\theta_1 \cos\theta_4 \cos\phi_3 - \lambda_1 \sin 2\theta_3 \cos\theta_1 \sin\theta_4 \cos(\phi_1+\phi_3+\phi_4) - \lambda_1 \sin 2\theta_2 \sin\theta_1 \cos\theta_4 \cos\phi_2 \tag{B7}$$

$$\dot{\phi}_2 = 2[E_b(t) + E_d(t)] - \sin^2\theta_1\dot{\phi}_1 - \sin^2\theta_4\dot{\phi}_4 - 2\cot 2\theta_2 \cos\theta_1 \sin\theta_4 \times [\lambda_{3r}\cos(\phi_2+\phi_4) + \lambda_{3i}\sin(\phi_2+\phi_4)] + 2\cot 2\theta_2 \cos\theta_4 \sin\theta_1 \times [\lambda_{3r}\cos(\phi_1+\phi_2) - \lambda_{3i}\sin(\phi_1+\phi_2)] - 2\lambda_1 \cot 2\theta_2 [\cos\theta_4 \cos\theta_1 \cos\phi_2 + \sin\theta_4 \sin\theta_1 \cos(\phi_1+\phi_2+\phi_4)] \tag{B8}$$

$$\dot{\phi}_3 = -2[E_b(t) + E_d(t)] - \sin^2\theta_1\dot{\phi}_1 - \sin^2\theta_4\dot{\phi}_4 + 2\cot 2\theta_3 \cos\theta_1 \sin\theta_4 \times [\lambda_{3r}\cos(\phi_3+\phi_4) + \lambda_{3i}\sin(\phi_3+\phi_4)] - 2\cot 2\theta_3 \cos\theta_4 \sin\theta_1 \times [\lambda_{3r}\cos(\phi_1+\phi_3) - \lambda_{3i}\sin(\phi_1+\phi_3)] + 2\lambda_1 \cot 2\theta_3 [\cos\theta_4 \cos\theta_1 \cos\phi_3 + \sin\theta_4 \sin\theta_1 \cos(\phi_1+\phi_3+\phi_4)] \tag{B9}$$

$$\dot{\phi}_4 = \frac{\csc 2\theta_4}{2\left(\cos^2\theta_2 - \sin^2\theta_3\right)}\{\sin 2\theta_2 \cos\theta_1 \cos\theta_4 \times [\lambda_{3r}\cos(\phi_2+\phi_4) + \lambda_{3i}\sin(\phi_2+\phi_4)] + \sin 2\theta_2 \sin\theta_1 \sin\theta_4 \times [\lambda_{3r}\cos(\phi_1+\phi_2) - \lambda_{3i}\sin(\phi_1+\phi_2)] - \sin 2\theta_3 \sin\theta_1 \sin\theta_4 \times [\lambda_{3r}\cos(\phi_1+\phi_3) - \lambda_{3i}\sin(\phi_1+\phi_3)] - \sin 2\theta_3 \cos\theta_1 \cos\theta_4 \times [\lambda_{3r}\cos(\phi_3+\phi_4) + \lambda_{3i}\sin(\phi_3+\phi_4)] + \lambda_1 \sin 2\theta_2 \cos\theta_4 \sin\theta_1 \cos(\phi_1+\phi_2+\phi_4) + \lambda_1 \sin 2\theta_3 \sin\theta_4 \cos\theta_1 \cos\phi_3 - \lambda_1 \sin 2\theta_3 \cos\theta_4 \sin\theta_1 \cos(\phi_1+\phi_3+\phi_4) - \lambda_1 \sin 2\theta_2 \sin\theta_4 \cos\theta_1 \cos\phi_2 \tag{B10}$$

$$\dot{\phi}_5 = \left[\chi_3^{(d)}(p,t) + \chi_3^{(g)}(p,t)\right] - \left[\chi_2^{(d)}(p,t) + \chi_2^{(g)}(p,t)\right] + 2\cot 2\theta_5 \left\{\cos\phi_5\left[\chi_7^{(d)}(p,t) + \chi_7^{(g)}(p,t)\right] - \sin\phi_5\left[\chi_8^{(d)}(p,t) + \chi_8^{(g)}(p,t)\right]\right\} \tag{B11}$$

$$\dot{\theta}_6 = \left[\chi_4^{(d)}(p,t) + \chi_4^{(g)}(p,t)\right] - \left[\chi_1^{(d)}(p,t) + \chi_1^{(g)}(p,t)\right] + 2\cot 2\theta_6 \left\{\cos\phi_6\left[\chi_5^{(d)}(p,t) + \chi_5^{(g)}(p,t)\right] - \sin\phi_6\left[\chi_6^{(d)}(p,t) + \chi_6^{(g)}(p,t)\right]\right\} \tag{B12}$$

where

$$\chi_1^{(d)} = \frac{1}{2}\cos 2\theta_2 [E_b + E_d] - \sin\theta_1 \cos\theta_4 \sin 2\theta_2 \times [\lambda_{3r}\cos(\phi_1+\phi_2) - \lambda_{3i}\sin(\phi_1+\phi_2)] + \sin\theta_4 \cos\theta_1 \sin 2\theta_2 \times [\lambda_{3r}\cos(\phi_2+\phi_4) + \lambda_{3i}\sin(\phi_2+\phi_4)] + \lambda_1 \cos\theta_1 \cos\theta_4 \sin 2\theta_2 \cos\phi_2 + \lambda_1 \sin\theta_1 \sin\theta_4 \sin 2\theta_2 \cos(\phi_1+\phi_2+\phi_4) \tag{B13}$$

$$\chi_1^{(g)} = -\sin^2\theta_1 \cos^2\theta_2 \dot{\phi}_1 + \sin^2\theta_2 \dot{\phi}_2 + \sin^2\theta_2 \sin^2\theta_4 \dot{\phi}_4 \tag{B14}$$

$$\chi_2^{(d)} = \frac{1}{2}\cos 2\theta_3 [E_b + E_d] + \sin\theta_4 \cos\theta_1 \sin 2\theta_3 \times [\lambda_{3r}\cos(\phi_3+\phi_4) + \lambda_{3i}\sin(\phi_3+\phi_4)] - \sin\theta_1 \cos\theta_4 \sin 2\theta_3 \times [\lambda_{3r}\cos(\phi_1+\phi_3) - \lambda_{3i}\sin(\phi_1+\phi_3)] + \lambda_1 \cos\theta_1 \cos\theta_4 \sin 2\theta_3 \cos\phi_3 + \lambda_1 \sin\theta_1 \sin\theta_4 \sin 2\theta_3 \cos(\phi_1+\phi_3+\phi_4) \tag{B15}$$

$$\chi_2^{(g)} = -\sin^2\theta_1 \cos^2\theta_3 \dot{\phi}_1 + \sin^2\theta_3 \dot{\phi}_3 + \sin^2\theta_3 \sin^2\theta_4 \dot{\phi}_4 \tag{B16}$$

$$\chi_3^{(d)} = -\chi_1^{(d)} \tag{B17}$$

$$\chi_3^{(g)} = -\chi_1^{(g)} \tag{B18}$$

$$\chi_4^{(d)} = -\chi_2^{(d)} \tag{B19}$$

$$\chi_4^{(g)} = -\chi_2^{(g)} \tag{B20}$$

$$\chi_5^{(d)} = \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 [\lambda_{3r}\cos(\phi_2+\phi_4-\phi_1-\phi_3) + \lambda_{3i}\sin(\phi_2+\phi_4-\phi_1-\phi_3)] + \lambda_1 \sin\theta_2 \sin\theta_3 [\sin\theta_1 \cos\theta_4 \cos(\phi_1-\phi_2+\phi_3) - \sin\theta_4 \cos\theta_1 \cos(\phi_2-\phi_3+\phi_4)] + \sin\theta_1 \cos\theta_2 \cos\theta_3 \sin\theta_4 \times [\lambda_{3r}\cos(\phi_1-\phi_4) - \lambda_{3i}\sin(\phi_1-\phi_4)] + \cos\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_4 \times [\lambda_{3r}\cos(\phi_2-\phi_3) - \lambda_{3i}\sin(\phi_2-\phi_3)] + \lambda_1 \cos\theta_2 \cos\theta_3 (\sin\theta_1 \cos\theta_4 \cos\phi_1 + \sin\theta_4 \cos\theta_1 \cos\phi_4) + \lambda_{3r}\cos\theta_1 \cos\theta_2 \cos\theta_3 \cos\theta_4 \tag{B21}$$

$$\chi_5^{(g)} = \dot{\theta}_1 \sin(\phi_1+\phi_3)\cos\theta_2\sin\theta_3 + \dot{\theta}_4 \sin(\phi_2+\phi_4)\sin\theta_2\cos\theta_3 + \frac{1}{2}\Big[\dot{\phi}_1\cos(\phi_1+\phi_3)\cos\theta_2\sin\theta_3\sin 2\theta_1 + \dot{\phi}_4\cos(\phi_2+\phi_4)\cos\theta_3\sin\theta_2\sin 2\theta_4\Big] \quad \text{(B22)}$$

$$\begin{aligned}\chi_6^{(d)} = & -\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 \\ & \times[\lambda_{3r}\sin(\phi_2+\phi_4-\phi_1-\phi_3) \\ & -\lambda_{3i}\cos(\phi_2+\phi_4-\phi_1-\phi_3)] \\ & +\lambda_1\sin\theta_2\sin\theta_3 \\ & \times[\sin\theta_1\cos\theta_4\sin(\phi_1-\phi_2+\phi_3) \\ & +\sin\theta_4\cos\theta_1\sin(\phi_2-\phi_3+\phi_4)] \\ & +\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4 \\ & \times[\lambda_{3r}\sin(\phi_1-\phi_4)+\lambda_{3i}\cos(\phi_1-\phi_4)] \\ & -\cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\ & \times[\lambda_{3r}\sin(\phi_2-\phi_3)+\lambda_{3i}\cos(\phi_2-\phi_3)] \\ & +\lambda_1\cos\theta_2\cos\theta_3 \\ & \times(\sin\theta_1\cos\theta_4\cos\phi_1+\sin\theta_4\cos\theta_1\cos\phi_4) \\ & -\lambda_{3i}\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\end{aligned} \quad \text{(B23)}$$

$$\chi_6^{(g)} = -\dot{\theta}_1 \cos(\phi_1+\phi_3)\cos\theta_2\sin\theta_3 + \dot{\theta}_4 \cos(\phi_2+\phi_4)\sin\theta_2\cos\theta_3 + \frac{1}{2}\Big[\dot{\phi}_1\sin(\phi_1+\phi_3)\cos\theta_2\sin\theta_3\sin 2\theta_1 - \dot{\phi}_4\sin(\phi_2+\phi_4)\cos\theta_3\sin\theta_2\sin 2\theta_4\Big] \quad \text{(B24)}$$

$$\begin{aligned}\chi_7^{(d)} = & \sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 \\ & \times[\lambda_{3r}\cos(\phi_1+\phi_2-\phi_3-\phi_4) \\ & -\lambda_{3i}\sin(\phi_1+\phi_2-\phi_3-\phi_4)] \\ & +\lambda_1\sin\theta_2\sin\theta_3 \\ & \times[\sin\theta_1\cos\theta_4\cos(\phi_1+\phi_2-\phi_3) \\ & -\sin\theta_4\cos\theta_1\cos(\phi_2-\phi_3-\phi_4)] \\ & +\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4 \\ & \times[\lambda_{3r}\cos(\phi_1-\phi_4)-\lambda_{3i}\sin(\phi_1-\phi_4)] \\ & +\cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\ & \times[\lambda_{3r}\cos(\phi_2-\phi_3)+\lambda_{3i}\sin(\phi_2-\phi_3)] \\ & +\lambda_1\cos\theta_2\cos\theta_3 \\ & \times(\sin\theta_1\cos\theta_4\cos\phi_1-\sin\theta_4\cos\theta_1\cos\phi_4) \\ & +\lambda_{3r}\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\end{aligned} \quad \text{(B25)}$$

$$\chi_7^{(g)} = -\dot{\theta}_1 \sin(\phi_1+\phi_2)\cos\theta_3\sin\theta_2 - \dot{\theta}_4 \sin(\phi_3+\phi_4)\sin\theta_3\cos\theta_2 - \frac{1}{2}\Big[\dot{\phi}_1\cos(\phi_1+\phi_2)\cos\theta_3\sin\theta_2\sin 2\theta_1 + \dot{\phi}_4\cos(\phi_3+\phi_4)\cos\theta_2\sin\theta_3\sin 2\theta_4\Big] \quad \text{(B26)}$$

$$\begin{aligned}\chi_8^{(d)} = & -\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4 \\ & \times[\lambda_{3r}\sin(\phi_1+\phi_2-\phi_3-\phi_4) \\ & -\lambda_{3i}\cos(\phi_1+\phi_2-\phi_3-\phi_4)] \\ & -\lambda_1\sin\theta_2\sin\theta_3 \\ & \times[\sin\theta_1\cos\theta_4\sin(\phi_1+\phi_2-\phi_3) \\ & -\sin\theta_4\cos\theta_1\sin(\phi_2-\phi_3-\phi_4)] \\ & -\sin\theta_1\cos\theta_2\cos\theta_3\sin\theta_4 \\ & \times[\lambda_{3r}\sin(\phi_1-\phi_4)+\lambda_{3i}\cos(\phi_1-\phi_4)] \\ & -\cos\theta_1\sin\theta_2\sin\theta_3\cos\theta_4 \\ & \times[\lambda_{3r}\sin(\phi_2-\phi_3)-\lambda_{3i}\cos(\phi_2-\phi_3)] \\ & -\lambda_1\cos\theta_2\cos\theta_3 \\ & \times(\sin\theta_1\cos\theta_4\cos\phi_1+\sin\theta_4\cos\theta_1\cos\phi_4) \\ & +\lambda_{3i}\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4\end{aligned} \quad \text{(B27)}$$

$$\chi_8^{(g)} = -\dot{\theta}_1 \cos(\phi_1+\phi_2)\cos\theta_3\sin\theta_2 + \dot{\theta}_4 \cos(\phi_3+\phi_4)\sin\theta_3\cos\theta_2 + \frac{1}{2}\Big[\dot{\phi}_1\sin(\phi_1+\phi_2)\cos\theta_3\sin\theta_2\sin 2\theta_1 - \dot{\phi}_4\sin(\phi_3+\phi_4)\cos\theta_2\sin\theta_3\sin 2\theta_4\Big] \quad \text{(B28)}$$

with λ_{3r}, λ_{3i} being the real part and imaginary part of λ_3, respectively.

The auxiliary equations for μ_m and v_m $(m = 1, 2 \ldots, 16)$ are

$$\dot{\mu}_1 = -l_{+sr}\sin v_1 - l_{+si}\cos v_1 - \lambda_1\mu_{13}\sin(v_1+v_{13}) - \mu_{16}[\lambda_{3r}\sin(v_1+v_{16})+\lambda_{3i}\cos(v_1+v_{16})] \quad \text{(B29)}$$

$$\dot{v}_1 = -E_b - \frac{1}{\mu_1}\{l_{+sr}\cos v_1 - l_{+si}\sin v_1 + \lambda_1\mu_{13}\cos(v_1+v_{13}) + \mu_{16}[\lambda_{3r}\cos(v_1+v_{16})-\lambda_{3i}\sin(v_1+v_{16})]\} \quad \text{(B30)}$$

$$\dot{\mu}_2 = -l_{-sr}\sin v_2 - l_{-si}\cos v_2 - \lambda_2\mu_{15}\sin(v_2+v_{15}) - \mu_{11}[\lambda_{4r}\sin(v_2+v_{11})+\lambda_{4i}\cos(v_2+v_{11})] \quad \text{(B31)}$$

$$\dot{v}_2 = -E_b - \frac{1}{\mu_2}\{l_{-sr}\cos v_2 - l_{-si}\sin v_2 + \lambda_2\mu_{15}\cos(v_2+v_{15}) + \mu_{11}[\lambda_{4r}\cos(v_2+v_{11})-\lambda_{4i}\sin(v_2+v_{11})]\} \quad \text{(B32)}$$

$$\dot{\mu}_3 = -\lambda_1\mu_{15}\sin(v_3+v_{15}) - \mu_{14}[\lambda_{3r}\sin(v_3+v_{14})+\lambda_{3i}\cos(v_3+v_{14})] \quad \text{(B33)}$$

$$\dot{v}_3 = -E_b - \frac{1}{\mu_3}\{\lambda_1\mu_{15}\cos(v_3+v_{15}) + \mu_{14}[\lambda_{3r}\cos(v_3+v_{14})-\lambda_{3i}\sin(v_3+v_{14})]\} \quad \text{(B34)}$$

$$\dot{\mu}_4 = -\lambda_2\mu_{16}\sin(v_4+v_{16}) -\mu_{13}\left[\lambda_{4r}\sin(v_4+v_{13})+\lambda_{4i}\cos(v_4+v_{13})\right] \quad \text{(B35)}$$

$$\dot{v}_4 = -E_b - \frac{1}{\mu_4}\{\lambda_2\mu_{16}\cos(v_4+v_{16}) +\mu_{13}\left[\lambda_{4r}\cos(v_4+v_{13})-\lambda_{4i}\sin(v_4+v_{13})\right]\} \quad \text{(B36)}$$

$$\dot{\mu}_5 = -l_{+sr}\sin v_5 - l_{+si}\cos v_5 -\lambda_1\mu_9\sin(v_9+v_5) -\mu_{12}\left[\lambda_{4r}\sin(v_{12}+v_5)+\lambda_{4i}\cos(v_{12}+v_5)\right] \quad \text{(B37)}$$

$$\dot{v}_5 = E_d - \frac{1}{\mu_5}\{l_{+sr}\cos v_5 - l_{+si}\sin v_5 +\lambda_1\mu_9\cos(v_9+v_5) +\mu_{12}\left[\lambda_{4r}\cos(v_{12}+v_5)-\lambda_{4i}\sin(v_{12}+v_5)\right]\} \quad \text{(B38)}$$

$$\dot{\mu}_6 = -l_{-sr}\sin v_6 - l_{-si}\cos v_6 -\lambda_2\mu_{10}\sin(v_6+v_{10}) -\mu_{11}\left[\lambda_{3r}\sin(v_{11}+v_6)+\lambda_{3i}\cos(v_{11}+v_6)\right] \quad \text{(B39)}$$

$$\dot{v}_6 = E_d - \frac{1}{\mu_6}\{l_{-sr}\cos v_6 - l_{-si}\sin v_6 +\lambda_2\mu_{10}\cos(v_{10}+v_6) +\mu_{11}\left[\lambda_{3r}\cos(v_{11}+v_6)-\lambda_{3i}\sin(v_{11}+v_6)\right]\} \quad \text{(B40)}$$

$$\dot{\mu}_7 = -\lambda_1\mu_{11}\sin(v_{11}+v_7) - \mu_{10}\left[\lambda_{4r}\sin(v_7+v_{10}) +\lambda_{3i}\cos(v_7+v_{10})\right] \quad \text{(B41)}$$

$$\dot{v}_7 = E_d - \frac{1}{\mu_7}\{\lambda_1\mu_{11}\cos(v_7+v_{11}) +\mu_{10}\left[\lambda_{4r}\cos(v_7+v_{10}) -\lambda_{4i}\sin(v_7+v_{10})\right]\} \quad \text{(B42)}$$

$$\dot{\mu}_8 = -\lambda_2\mu_{12}\sin(v_8+v_{12}) -\mu_9\left[\lambda_{3r}\sin(v_9+v_8)+\lambda_{3i}\cos(v_9+v_8)\right] \quad \text{(B43)}$$

$$\dot{v}_8 = E_d - \frac{1}{\mu_8}\{\lambda_2\mu_{12}\cos(v_{12}+v_8) +\mu_9\left[\lambda_{3r}\cos(v_9+v_8)-\lambda_{3i}\sin(v_8+v_9)\right]\} \quad \text{(B44)}$$

$$\dot{\mu}_9 = -\lambda_1\mu_{15}\sin(v_9+v_5) - \mu_8\left[\lambda_{3r}\sin(v_9+v_8) +\lambda_{3i}\cos(v_9+v_8)\right] \quad \text{(B45)}$$

$$\dot{v}_9 = -E_b - \frac{1}{\mu_9}\{\lambda_1\mu_5\cos(v_9+v_5) +\mu_8\left[\lambda_{3r}\cos(v_9+v_8)-\lambda_{3i}\sin(v_8+v_9)\right]\} \quad \text{(B46)}$$

$$\dot{\mu}_{10} = -\lambda_2\mu_6\sin(v_{10}+v_6) - \mu_7\left[\lambda_{4r}\sin(v_{10}+v_7) +\lambda_{4i}\cos(v_{10}+v_7)\right] \quad \text{(B47)}$$

$$\dot{v}_{10} = -E_b - \frac{1}{\mu_{10}}\{\lambda_2\mu_6\cos(v_{10}+v_6) +\mu_7\left[\lambda_{4r}\cos(v_{10}+v_7) -\lambda_{4i}\sin(v_{10}+v_7)\right]\} \quad \text{(B48)}$$

$$\dot{\mu}_{11} = -\lambda_1\mu_7\sin(v_{11}+v_7) - \mu_6\left[\lambda_{3r}\sin(v_{11}+v_6) +\lambda_{3i}\cos(v_{11}+v_6)\right] \quad \text{(B49)}$$

$$\dot{v}_{11} = -E_b - \frac{1}{\mu_{11}}\{\lambda_1\mu_7\cos(v_{11}+v_7) +\mu_6\left[\lambda_{3r}\cos(v_{11}+v_6) -\lambda_{3i}\sin(v_{11}+v_6)\right]\} \quad \text{(B50)}$$

$$\dot{\mu}_{12} = -\lambda_2\mu_8\sin(v_{12}+v_8) - \mu_5\left[\lambda_{4r}\sin(v_{12}+v_5) +\lambda_{4i}\cos(v_{12}+v_5)\right] \quad \text{(B51)}$$

$$\dot{v}_{12} = -E_b - \frac{1}{\mu_{12}}\{\lambda_2\mu_8\cos(v_{12}+v_8) +\mu_5\left[\lambda_{4r}\cos(v_{12}+v_5) -\lambda_{4i}\sin(v_{12}+v_5)\right]\} \quad \text{(B52)}$$

$$\dot{\mu}_{13} = -\lambda_1\mu_1\sin(v_{13}+v_1) - \mu_4\left[\lambda_{4r}\sin(v_{13}+v_4) +\lambda_{4i}\cos(v_{13}+v_4)\right] \quad \text{(B53)}$$

$$\dot{v}_{13} = E_d - \frac{1}{\mu_{13}}\{\lambda_1\mu_1\cos(v_{13}+v_1) +\mu_4\left[\lambda_{4r}\cos(v_{13}+v_4) -\lambda_{4i}\sin(v_{13}+v_4)\right]\} \quad \text{(B54)}$$

$$\dot{\mu}_{14} = -\lambda_2\mu_2\sin(v_{14}+v_2) - \mu_3\left[\lambda_{3r}\sin(v_3+v_{14}) +\lambda_{3i}\cos(v_{14}+v_3)\right] \quad \text{(B55)}$$

$$\dot{v}_{14} = E_d - \frac{1}{\mu_{14}}\{\lambda_2\mu_2\cos(v_{14}+v_2) +\mu_3\left[\lambda_{3r}\cos(v_3+v_{14}) -\lambda_{3i}\sin(v_3+v_{14})\right]\} \quad \text{(B56)}$$

$$\dot{\mu}_{15} = -\lambda_1\mu_3\sin(v_{15}+v_3) - \mu_2\left[\lambda_{4r}\sin(v_{15}+v_2) +\lambda_{4i}\cos(v_{15}+v_2)\right] \quad \text{(B57)}$$

$$\dot{v}_{15} = E_d - \frac{1}{\mu_{15}}\{\lambda_1\mu_3\cos(v_{15}+v_3) +\mu_2\left[\lambda_{4r}\cos(v_{15}+v_2) -\lambda_{4i}\sin(v_{15}+v_2)\right]\} \quad \text{(B58)}$$

$$\dot{\mu}_{16} = -\lambda_2\mu_4 \sin(v_{16}+v_4) - \mu_1 [\lambda_{3r} \sin(v_1+v_{16}) + \lambda_{3i} \cos(v_1+v_{16})] \quad \text{(B59)}$$

$$\dot{v}_{16} = E_d - \frac{1}{\mu_{16}} \{\lambda_2\mu_4 \cos(v_{16}+v_4) + \mu_1 [\lambda_{3r} \cos(v_1+v_{16}) - \lambda_{3i} \sin(v_1+v_{16})]\} \quad \text{(B60)}$$

where λ_{3r}, λ_{3i}, λ_{4r}, and λ_{4i} are the real and imaginary parts of λ_3, λ_4, respectively.

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